Laser cooling of two trapped ions: Sideband cooling beyond the Lamb-Dicke limit

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We study laser cooling of two ions that are trapped in a harmonic potential and interact by Coulomb repulsion. Sideband cooling in the Lamb-Dicke regime is shown to work analogously to sideband cooling of a single ion. Outside the Lamb-Dicke regime, the incommensurable frequencies of the two vibrational modes result in a quasicontinuous energy spectrum that significantly alters the cooling dynamics. The cooling time decreases nonlinearly with the linewidth of the cooling transition, and the effect of dark states which may slow down the cooling is considerably reduced. We show that cooling to the ground state is also possible outside the Lamb-Dicke regime. We develop the model and use quantum Monte Carlo calculations for specific examples. We show that a rate equation treatment is a good approximation in all cases. [S1050-2947(99)11605-6]

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I. INTRODUCTION

The emergence of schemes that utilize trapped ions or atoms for quantum information, and the interest in quantum statistics of ultracold atoms, have provided renewed interest and applications for laser cooling techniques [1]. The present goal is to laser cool several atoms to a pure quantum state (to the motional ground state), and experimental [2] and theoretical efforts [3] are made in this direction. The cooling of a large number of particles using lasers is a prerequisite for coherent control of atomic systems [4,5]. In quantum information, for example, laser cooling to the motional ground state is a fundamental step in the preparation of trapped ions for quantum logic [5]. Coherent control and manipulation of information requires that each ion be individually addressable with a laser [6], and thus restricts the choice of the trap frequency, and consequently the regime in which cooling must work, to relatively shallow traps [7].

Laser cooling of single ions in traps has been extensively studied [8], and in particular sideband cooling has been demonstrated to be a successful technique for cooling single ions to the ground state of a harmonic trap [9]. Sufficient conditions for sideband cooling a two level system are (i) the radiative linewidth γ is smaller than the trap frequency ν, such that motional sidebands, i.e., optical transitions that involve the creation or annihilation of a specific number of motional quanta, can be selectively excited; and (ii) the Lamb-Dicke limit is fulfilled, i.e., the ion's motional excursion is much smaller than the laser wavelength. The first condition can be achieved through an adequate choice of atomic transition or a manipulation of the internal atomic structure [10], while the Lamb-Dicke regime requires the trap frequency to be much larger than the recoil frequency of the optical transition.

For more than one ion, as required in quantum logic schemes, individual addressing imposes small trap frequencies, whereas sideband cooling imposes high trap frequencies. Furthermore, the Coulomb interaction between the particles makes the problem much more complex, and it is not obvious whether the techniques developed for single ions can be transferred directly to this situation. Experimentally, sideband cooling of two ions to the ground state has been achieved in a Paul trap that operates in the Lamb-Dicke limit [11]. However in this experiment the Lamb-Dicke regime required such a high trap frequency that the distance between the ions does not allow their individual addressing with a laser. For this purpose, and also for an extension beyond two ions, linear ion traps [12,13] are the most suitable systems, and for their physical parameters laser cooling to the ground state is a goal yet to be achieved. Laser cooling of two ions into the ground state is the problem that we address in this paper.

Theoretical studies on cooling of single ions outside the Lamb-Dicke regime exist [14,15], while laser cooling of more than one ion has been analyzed only in the Lamb-Dicke regime [16,17]. In this paper, we investigate laser cooling, as developed for single ions, when it is applied to two or more ions. We focus our attention on sideband cooling, showing and discussing new physical effects which arise because of the presence of two interacting particles. Doppler cooling will be discussed in a future work. We show that cooling of two ions outside the Lamb-Dicke regime presents novel features with respect to single-ion cooling, and we show how the preparation of the two ions in a pure quantum state is possible. The results will allow us to obtain some insight into the more general problem of cooling a string of N ions. This is not only relevant for quantum logic with N-ion strings but also for laser cooling of ion clusters in Paul and Penning traps [13,18].

The paper is organized as follows. In Sec. II we introduce and discuss the model which we will use throughout the paper, and discuss some concepts developed in sideband cooling of one ion in relation to the presence of more than one ion. In Sec. III we study and discuss sideband cooling of two ions inside and outside the Lamb-Dicke regime, and compare the two different behaviors. Finally, in the conclusions we summarize the main results, and discuss the problem of cooling $N>2$ ions.

II. MODEL

We consider two ions of mass m and charge e placed in a one-dimensional harmonic potential of frequency ν. We as-

sume the ions to be strongly trapped in the other spatial dimensions, so that their motion in those directions is frozen out. Their internal structure is described by a two-level system with ground state $|g\rangle$, excited state $|e\rangle$, and resonance frequency ω_0. The ions interact with laser light at frequency ω_L and wave vector k. For classical laser light and in the rotating-wave approximation, the Hamiltonian of the system is

$$H=H_i+H_{\rm mec}+V. \tag{1}$$

Here H_i is the internal energy in the rotating frame,

$$H_i=-\delta\sum_{j=1,2}|e\rangle_j\ {}_j\langle e|, \tag{2}$$

where $\delta=\omega_L-\omega_0$ is the detuning, j labels the ion ($j=1$ and 2), and we have taken $\hbar=1$. $H_{\rm mec}$ is the mechanical Hamiltonian,

$$H_{\rm mec}=\frac{p_1^2}{2m}+\frac{p_2^2}{2m}+\frac{1}{2}m\nu^2x_1^2+\frac{1}{2}m\nu^2x_2^2+\frac{e^2}{4\pi\epsilon_0|x_1-x_2|}, \tag{3}$$

with x_j and p_j the position and momentum of the jth ion ($j=1$ and 2), and V describes the interaction between laser and atoms,

$$V=\sum_{j=1,2}\frac{\Omega(x_j)}{2}[\sigma_j^\dagger e^{-ikx_j\cos\theta}+\text{H.c.}]. \tag{4}$$

Here $\Omega(x_j)$ is the Rabi frequency at the position x_j; $\sigma_j^\dagger$ and σ_j are the raising and lowering dipole operators, respectively, defined on the jth ion ($j=1$ and 2); and θ is the angle between the laser wave vector and the trap axis.

Using the center-of-mass (COM) and relative coordinates, the mechanical Hamiltonian in Eq. (3) is composed of two separate terms: one for the COM motion which describes a particle of mass $M=2m$ interacting with a harmonic potential of frequency ν; the other represents the relative motion which describes a particle of mass $\mu=m/2$ interacting with a potential, which is the sum of a harmonic potential of frequency ν and a Coulomb-type central potential. This potential may be approximated by a harmonic-oscillator potential of frequency $\nu_r=\sqrt{3}\nu$, obtained through the truncation at the second order of its Taylor expansion around the equilibrium distance $x_0=(2e^2/4\pi\epsilon_0M\nu)^{1/3}$ between the ions [16,17]. In Appendix A we discuss this approximation and we show that it is valid in the regime that we are going to study. With this approximation term (3) becomes (apart from a constant)

$$H_{\rm mec}=\frac{P^2}{2M}+\frac{1}{2}M\nu^2X^2+\frac{p^2}{2\mu}+\frac{1}{2}\mu\nu_r^2x^2, \tag{5}$$

where $X=(x_1+x_2)/2$, $P=p_1+p_2$ are the position and momentum of the COM, respectively, and $x=x_1-x_2-x_0$, $p=(p_1-p_2)/2$ are the position and momentum of the relative motion. Thus term (5), apart from a constant, can be rewritten as

$$H_{\rm mec}=\nu a_0^\dagger a_0+\nu_r a_r^\dagger a_r, \tag{6}$$

where we have defined $X=\sqrt{1/2M\nu_0}(a_0^\dagger+a_0)$, $P=i\sqrt{M\nu_0/2}(a_0^\dagger-a_0)$, $x=x_0+\sqrt{1/2\mu\nu_r}(a_r^\dagger+a_r)$, and $p=i\sqrt{\mu\nu_r/2}(a_r^\dagger-a_r)$, with a_0 and $a_0^\dagger$ the annihilation and creation operators for the COM mode, respectively, and a_r and $a_r^\dagger$ the corresponding ones for the relative motion (stretch) mode. We stress that in this representation the mechanical problem of two ions interacting through Coulomb forces is reduced to the one of two harmonic oscillators, while the interaction of each ion with the radiation is now transformed into a nonlinear coupling between the harmonic oscillators. In general, N ions in a trap can be described by a set of N harmonic oscillators, coupled by laser light [19].

The master equation for the density matrix ρ of the two-ion system is

$$\frac{d}{dt}\rho=-\frac{i}{\hbar}[H,\rho]+L\rho. \tag{7}$$

Here L is the Liouvillian describing the incoherent evolution of the system:

$$L\rho=\frac{\gamma}{2}\sum_{j=1,2}[2\sigma_j\tilde{\rho}_j\sigma_j^\dagger-\sigma_j^\dagger\sigma_j\rho-\rho\sigma_j^\dagger\sigma_j], \tag{8}$$

where γ is the decay rate out of the internal excited state $|e\rangle$, and $\tilde{\rho}_j$ describes the density matrix after a spontaneous emission for the jth ion:

$$\tilde{\rho}_j=\int_{-1}^{1}du\,N(u)e^{ikux_j}\rho e^{-ikux_j}, \tag{9}$$

with $N(u)$ being the dipole pattern for the decay.

In this treatment we have neglected both dipole-dipole interaction between the ions and quantum statistical properties. This approximation is justified in the regime that we investigate, which is characteristic of experiments using linear ion traps for quantum information [12]. In those traps the equilibrium distance between the ions is of the order of 10μm, while the laser wavelength is typically in the visible region and the individual ionic wave packets have spatial widths of the order of 10–100 nm. From these considerations we can consider the two ions in a linear trap as two distinguishable particles [20] in a harmonic potential which interact solely with Coulomb forces. On the basis of these considerations, we will use Eq. (7) for the numerical simulations presented below.

For the following discussion it is instructive to look at the set of equations which one obtains from Eq. (7) in the limit of low saturation $\Omega\ll\gamma$, when the excited state $|e\rangle$ can be eliminated in second-order perturbation theory (we provide a detailed derivation of the equations in Appendix B). In the basis of states $|g,\mathbf{n}\rangle$, where $\mathbf{n}=(n_0,n_r)$ is a vector with COM vibrational number n_0 and stretch mode vibrational number n_r, we have the following set of equations:

$$\frac{d}{dt}\langle g,\mathbf{n}|\rho|g,\mathbf{m}\rangle = -i(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\nu}\,\langle g,\mathbf{n}|\rho|g,\mathbf{m}\rangle$$
$$+ i\frac{\Omega^2}{4}\sum_{\mathbf{k},\mathbf{l}}\left[\frac{\langle\mathbf{n}|e^{ikx_1\cos\theta}|\mathbf{k}\rangle\langle\mathbf{k}|e^{-ikx_1\cos\theta}|\mathbf{l}\rangle}{(\mathbf{k}-\mathbf{l})\cdot\boldsymbol{\nu}-\delta-i\gamma/2}\langle g,\mathbf{l}|\rho|g,\mathbf{m}\rangle - \langle g,\mathbf{n}|\rho|g,\mathbf{l}\rangle\frac{\langle\mathbf{l}|e^{ikx_1\cos\theta}|\mathbf{k}\rangle\langle\mathbf{k}|e^{-ikx_1\cos\theta}|\mathbf{m}\rangle}{(\mathbf{k}-\mathbf{l})\cdot\boldsymbol{\nu}-\delta+i\gamma/2}\right]$$
$$+\frac{\Omega^2}{4}\sum_{\mathbf{k},\mathbf{j},\mathbf{r},\mathbf{s}}\int_{-1}^{1}du\,N(u)\langle\mathbf{n}|e^{iukx_1}|\mathbf{k}\rangle\langle\mathbf{k}|e^{-ikx_1\cos\theta}|\mathbf{r}\rangle\langle\mathbf{s}|e^{ikx_1\cos\theta}|\mathbf{j}\rangle\langle\mathbf{j}|e^{-iukx_1}|\mathbf{m}\rangle\langle g,\mathbf{r}|\rho|g,\mathbf{s}\rangle$$
$$\times\left[\frac{1}{((\mathbf{j}-\mathbf{s}-\mathbf{k}+\mathbf{r})\cdot\boldsymbol{\nu}+i\gamma)((\mathbf{j}-\mathbf{s})\cdot\boldsymbol{\nu}-\delta-i\gamma/2)}+\frac{1}{((\mathbf{k}-\mathbf{r}-\mathbf{j}+\mathbf{s})\cdot\boldsymbol{\nu}-i\gamma)((\mathbf{k}-\mathbf{r})\cdot\boldsymbol{\nu}-\delta+i\gamma/2)}\right], \tag{10}$$

where $\boldsymbol{\nu}=(\nu,\nu_r)$, and where for simplicity we have assumed that only ion 1 is illuminated, i.e., $\Omega(x_1)=\Omega$, $\Omega(x_2)=0$. This implies that we can address the ions individually with a well-focused laser beam. It also corresponds, for example, to a situation where the two ions are two different isotopes, of which only one is resonant with light [21].

When treating laser cooling in a harmonic trap, an important dimensionless quantity is the Lamb-Dicke parameter η, which for a single ion of mass m in a trap of frequency ν, interacting with laser light of wave vector k, is

$$\eta = k\sqrt{\frac{1}{2m\nu}} = \sqrt{\frac{\omega_{\rm rec}}{\nu}}, \tag{11}$$

where $\omega_{\rm rec}=k^2/2m$ is the recoil frequency. The parameter η appears in the kick operator $\exp(ikx)$ in the term describing the exchange of momentum between radiation and atoms, which, using the relation $x=\sqrt{1/2m\nu}(a^\dagger+a)$ and definition (11), is rewritten as $\exp(ikx)=\exp(i\eta(a^\dagger+a))$. The Lamb-Dicke regime corresponds to the condition $\sqrt{n}\,\eta\ll 1$, with n vibrational number; in other words, to the situation in which, during a spontaneous emission, a change in the vibrational number of the atomic state is unlikely due to energy conservation. In this regime the kick operator may be expanded in powers of η, and with good approximation the expansion may be truncated at the first order [8]. Another important parameter, as known from cooling of single ions, is the ratio between the radiative linewidth γ and the trap frequency ν: in the so-called strong confinement regime $\gamma/\nu\ll 1$ the laser can selectively excite sidebands of the optical transition which involve a well-defined change of the vibrational number n. In this regime, together with the Lamb-Dicke regime, sideband cooling works efficiently: when the laser is red detuned with $\delta=-\nu$, the system is cooled by approximately one phonon of energy ν in each fluorescence cycle, finally reaching the vibrational ground state $n=0$ [9]. In contrast, in the weak confinement regime $\gamma/\nu\geqslant 1$, transitions which involve different changes of the vibrational number n are excited simultaneously. This is the Doppler cooling regime, where the achievable minimum energy for a single ion is approximately $\gamma/2$ for a detuning $\delta=-\gamma/2$ [22].

Having introduced these basic concepts and methods of laser cooling of single ions in traps, we now return to the problem of two ions, to discuss how those techniques may be applied, and whether the same concepts are still valid. For N harmonic-oscillator modes we can define a Lamb-Dicke parameter for each mode in an analogous way to Eq. (11). For our case, $N=2$, the Lamb-Dicke parameters η_0 for the COM mode and η_r for the stretch mode are defined as

$$\eta_0 = k\sqrt{\frac{\hbar}{2M\nu}} = \frac{\eta}{\sqrt{2}},$$
$$\eta_r = \frac{k}{2}\sqrt{\frac{\hbar}{2\mu\nu_r}} = \frac{\eta}{\sqrt{2\sqrt{3}}}, \tag{12}$$

so that the kick operator for the jth ion ($j=1$ and 2) is written as

$$e^{ikx_j} = e^{i\eta_0(a_0^\dagger+a_0)}e^{(-1)^{j-1}i\eta_r(a_r^\dagger+a_r)}. \tag{13}$$

In general, for N ions $\eta_0=\eta/\sqrt{N}$ [19]. In the following, when we refer to the Lamb-Dicke regime, we will consider the situation where the conditions $\sqrt{n_0}\,\eta_0\ll 1$ and $\sqrt{n_r}\,\eta_r\ll 1$ are fulfilled. From Eq. (13) we see that in Eq. (4) the Lamb-Dicke parameters appear multiplied by the factor $\cos\theta$. Therefore, the Lamb-Dicke parameters for the coherent excitation, $\eta_i\cos\theta$, are always less than or equal to the Lamb-Dicke parameters defined in Eq. (12), which characterize the spontaneous emission.

To discuss the importance of the ratio γ/ν in the case of two ions, we first consider the bare spectrum of energies of our system with frequencies ν and $\sqrt{3}\nu$. In Fig. 1 we plot the

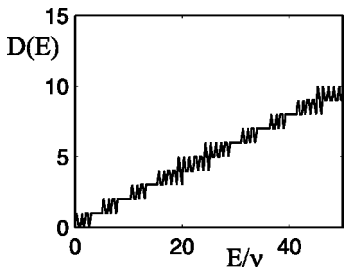

FIG. 1. Number of states $D(E)$ in the energy interval $[E,E+\delta E]$ plotted as a function of energy E in units of ν. The grid is $\delta E=\nu/3$.

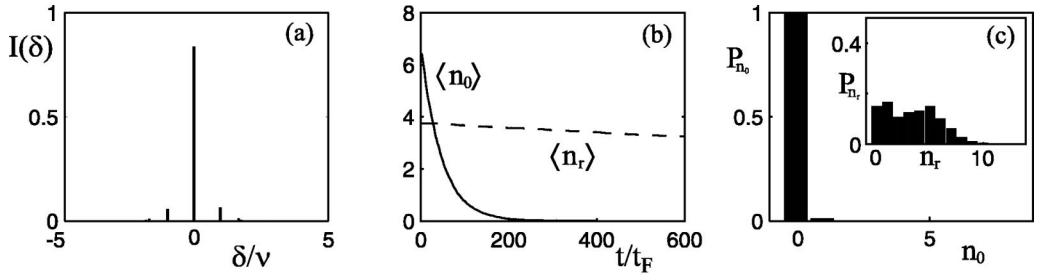

FIG. 2. (a) Absorption spectrum $I(\delta)$ vs detuning δ (in units of ν) for $\eta_0=0.1$ for a thermal distribution with average energy per mode $\bar{n}\nu=7.5\nu$, plotted on a grid of width $\nu/10$. (b) Plot of the average COM mode vibrational number $\langle n_0\rangle$ (solid line) and average stretch mode vibrational number $\langle n_r\rangle$ (dashed line) vs time in units $t_F=2\gamma/\Omega^2$, for $\eta_0=0.1$, $\gamma=0.2\nu$, $\Omega=0.034\nu$, $\delta=-\nu$, and atoms initially in a flat distribution on the states with energy $E\leqslant 15\nu$. (c) Population of the COM mode P_{n_0} (onset) and of stretch mode P_{n_r} (inset) vs the respective vibrational state number at $t=600t_F$.

function $D(E)$ vs the energy E, defined as the number of states in the interval of energy $[E,E+\delta E]$. From this figure we see that due to the *incommensurate* character of the frequencies the spectrum does not exhibit a well-distinguished series of energy levels, but rather tends toward a quasicontinuum. Therefore, the strong confinement requirement for sideband cooling needs to be reconsidered. The main question which we will address in the following is whether it is still possible to cool one mode to the ground state by means of sideband cooling. As we will show, the Lamb-Dicke parameter distinguishes two regimes which exhibit dramatic differences.

III. SIDEBAND COOLING OF TWO IONS

In the following we study sideband cooling of two ions, first in the Lamb-Dicke regime and then outside of this regime. We will show that in this latter case two-ion effects appear due to the dense spectrum of energy levels. In our calculations we first consider sideband cooling when laser light excites only one of the two ions directly. Thus in Eq. (4) we take $\Omega(x_1)=\Omega$, and $\Omega(x_2)=0$. Afterwards we compare this case to the one in which both are driven by light, i.e., $\Omega(x_1)=\Omega(x_2)=\Omega$, showing that the only difference between the two cases is the cooling time, which in the latter case scales by a factor $\frac{1}{2}$. In the following we assume that the laser wave vector is parallel to the trap axis, i.e., $\cos\theta=1$. This assumption facilitates the analysis and it is justified by the simple scaling just described. Furthermore, it corresponds to the case in which the two ions are two different ionic isotopes, of which one is driven by light [21]. At the end of this section we will briefly discuss cooling of two identical ions when the wave vector is not parallel to the trap axis.

A. Lamb-Dicke regime

In the Lamb-Dicke regime the Franck-Condon coefficients $\langle\mathbf{n}|\exp(ikx)|\mathbf{l}\rangle$ in the numerators of the right-hand side terms of Eq. (10) may be expanded in terms of the Lamb-Dicke parameters η_0 and η_r. The response of the system to laser light is governed by its absorption spectrum $I(\delta)$, which is evaluated by summing all contributions to laser-excited transitions at frequency ω_L,

$$I(\delta)=\sum_{(\mathbf{n}-\mathbf{l})\cdot\boldsymbol{\nu}=\delta}|\langle\mathbf{n}|\exp(ikx)|\mathbf{l}\rangle|^2P(\mathbf{n}), \tag{14}$$

where $P(\mathbf{n})$ is a normalized distribution of the states $|\mathbf{n}\rangle$. In the Lamb-Dicke regime, we find that $I(\delta)$ exhibits two main pairs of sidebands around the optical frequency ω_0: one at frequencies $\omega_0\pm\nu_0$ corresponding to the transition $n_0\rightarrow n_0\pm 1$, the other at frequencies $\omega_0\pm\nu_r$ corresponding to $n_r\rightarrow n_r\pm 1$ [see Fig. 2(a)]. The strength of these sidebands relative to the carrier $\mathbf{n}\rightarrow\mathbf{n}$ are proportional to η_0^2 and η_r^2, respectively. All the other sidebands have strengths of higher orders in η_0^2 and η_r^2. This implies that by selecting one of these four sidebands by laser excitation we will induce the corresponding phononic transition; for example by choosing the sideband corresponding to $(n_0,n_r)\rightarrow(n_0-1,n_r)$, we can cool the COM mode to its vibrational ground state, as for a single ion. This has been experimentally demonstrated by the NIST group at Boulder [11]. In Fig. 2 we plot the results of a quantum Monte Carlo (QMC) wave-function simulation [23] of Eq. (7) for two ions in a trap with Lamb-Dicke parameter $\eta_0=0.1$, radiative linewidth $\gamma=0.2\nu$, detuning $\delta=-\nu$ and an initially flat distribution for the states with energy $E\leqslant 15\nu$. In Fig. 2(b) the average vibrational numbers of the COM mode (solid line) and of the stretch mode (dashed line) are plotted as a function of time in unit of fluorescence cycles $t_F=2\gamma/\Omega^2$. The system behaves as if the two modes

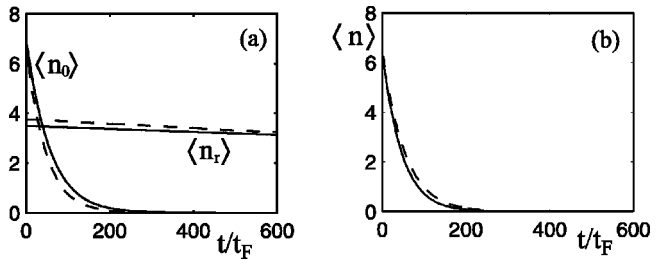

FIG. 3. (a) Comparison between the rate equation (solid line) and QMC calculation (dashed line). Same parameters as in Fig. 2(b). (b) Comparison between the time dependence of the COM average vibrational number as in Fig. 2(b) (solid line) and the average vibrational number for the case in which a single ion is cooled (dashed line). For the single ion the mass has been rescaled so that $\eta^{(1)}=\eta_0$, $\nu^{(1)}=\nu$, $\gamma=0.2\nu$, and $\Omega=0.034\nu$, with an initially flat distribution for the first 15 states.

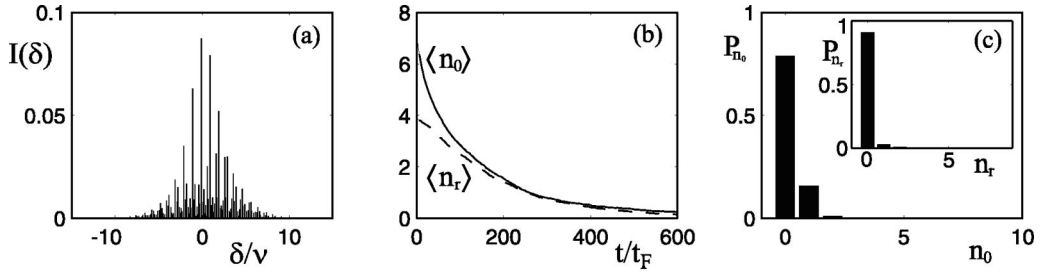

FIG. 4. (a) Absorption spectrum $I(\delta)$ vs detuning δ (in units of ν) for $\eta_0=0.6$ of ions in a thermal distribution with average energy per mode $\bar{n}\nu=7.5\nu$, plotted on a grid of width $\nu/10$. (b) Plot of $\langle n_0\rangle$ (solid line) and $\langle n_r\rangle$ (dashed line) vs time in unit $t_F=2\gamma/\Omega^2$, for $\eta_0=0.6$, $\gamma=0.2\nu$, $\Omega=0.034\nu$, $\delta=-2\nu$, and atoms initially in a flat distribution for states with energy $E\leqslant 15\nu$. (c) Population of the COM mode P_{n_0} (onset) and the stretch mode P_{n_r} (inset) vs the respective vibrational number state at $t=600t_F$.

were decoupled, since only one mode is cooled while the other remains almost frozen. Nevertheless the stretch mode is cooled on a much longer time scale, as an effect of off-resonant excitation. In Fig. 2(c) the populations of the vibrational states of the two modes are plotted at time $t=600t_F$, showing the COM mode in the ground state and the nearly uncooled stretch mode. In this limit we can neglect the coupling of the population to the coherences in Eq. (10), thus reducing the density-matrix equation in the low saturation limit to rate equations. In fact the coherences in Eq. (10) have either an oscillation frequency which is much larger than the fluorescence rate $1/t_F$, or a coupling to the population which is of higher order in the Lamb-Dicke parameter expansion, or both. Therefore, they can be neglected in the equations of the populations, and we obtain the set of rate equations

$$\frac{d}{dt}\langle \mathbf{n}|\rho|\mathbf{n}\rangle = -\gamma\frac{\Omega^2}{4}\sum_{\mathbf{k}}\frac{|\langle \mathbf{n}|e^{ikx_1}|\mathbf{k}\rangle|^2}{[(\mathbf{k}-\mathbf{n})\cdot\boldsymbol{\nu}-\delta]^2+\gamma^2/4}\langle \mathbf{n}|\rho|\mathbf{n}\rangle + \gamma\frac{\Omega^2}{4}\sum_{\mathbf{k},\mathbf{r}}\int_{-1}^{1}du\,N(u)\frac{|\langle \mathbf{n}|e^{ikux_1}|\mathbf{k}\rangle|^2|\langle \mathbf{k}|e^{-ikx_1}|\mathbf{r}\rangle|^2}{[(\mathbf{k}-\mathbf{r})\cdot\boldsymbol{\nu}-\delta]^2+\gamma^2/4}\langle \mathbf{r}|\rho|\mathbf{r}\rangle, \tag{15}$$

where we have omitted the label g of the states. The validity of this approximation is shown for the above case in Fig. 3(a), where the results of Fig. 2(b) are compared with those of a rate equation simulation according to Eq. (15). As a further proof that the two modes can be considered decoupled during the time in which the COM motion is cooled, in Fig. 3(b) we compare the time dependence of the average vibrational number of the COM mode with the one of a single trapped ion which is cooled under the same Lamb-Dicke parameter, radiative linewidth, trap frequency, Rabi frequency, and initial distribution as the COM mode. We see that the two curves overlap appreciably, justifying the picture of sideband cooling of two ions in the Lamb-Dicke regime as if the modes were decoupled from one another.

B. Outside the Lamb-Dicke regime

To illustrate the physical features of the system outside the Lamb-Dicke regime, in Fig. 4(a) we plot the absorption spectrum $I(\delta)$ as defined in Eq. (14) for two ions in a harmonic trap with $\eta_0=0.6$. We see that the spectrum exhibits many sidebands whose density increases as the detuning increases. The main consequence is that we cannot select a given sideband by choosing the laser frequency, but rather excite a group of resonances that correspond to transitions to a set of quasidegenerate states. The range of transitions that are excited increases with γ. In Figs. 4(b) and 4(c) we consider sideband cooling for Lamb-Dicke parameter $\eta_0=0.6$ and detuning $\delta=-2\nu$ [14], where the other parameters are the same as in Figs. 2(b) and 2(c). As one can see, the two modes are coupled and cooled together. Thus as a first big difference with respect to single-ion cooling, we see that here the energy is not taken away from one mode only; rather, it is subtracted from the system as a whole. Another striking difference appears in the cooling time, which is significantly longer in comparison with the time necessary to cool one single ion outside the Lamb-Dicke regime [14]. This slowing down is partly due to the increase of the dimension of the phase space where the cooling takes place: the presence of two modes makes the problem analogous to cooling in a two-dimensional trap, whose axes are coupled by the laser. The ions thus make a random walk in a larger phase space, and the cooling becomes slower. However, the cooling time is even considerably longer than one would expect taking the dimensionality into account. This can be explained by looking again at the spectrum in Fig. 4(a): despite the high density of resonances, the coupling between the states is still governed by the Franck-Condon coupling, i.e., by the terms in the numerator of Eq. (10) which outside the Lamb-Dicke limit oscillate with the vibrational numbers of the states. In the limit of linewidth $\gamma\ll\nu$, where a single sideband can be

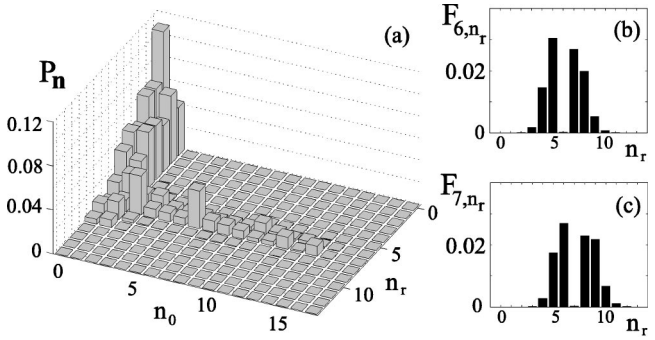

FIG. 5. (a) Population $P_{\mathbf{n}}=\langle \mathbf{n}|\rho|\mathbf{n}\rangle$ as a function of n_0 and n_r at a time $t=1000t_F$ and for $\gamma=0.02\nu$, $\Omega=0.17\gamma=0.0034\nu$, and $t_F=2\gamma/\Omega^2=3460/\nu$. All the other physical parameters are the same as in Figs. 4(b) and 4(c). (b) and (c) Modulus square of the Franck-Condon coefficients for the relative motion $F_{l,n_r}=|\langle l|e^{i\eta_r(a_r^\dagger+a_r)}|n_r\rangle|^2$ with $l=6$ and 7.

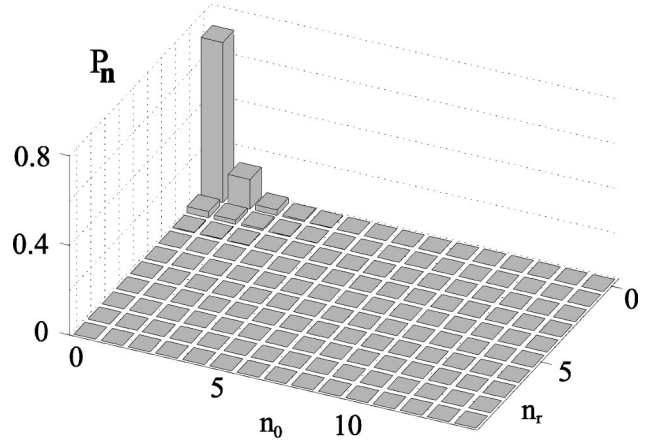

FIG. 6. Population $P_{\mathbf{n}}$ as a function of n_0 and n_r at a time $t=600t_F$ and for $\gamma=\nu$, $\Omega=0.17\gamma$, and $t_F=2\gamma/\Omega^2=69/\nu$. All other physical parameters are the same as in Figs. 4(b) and 4(c).

selected, we may encounter *dark states* like in cooling of single ions [14], i.e., states whose coupling to the resonantly excited state is very small since their motional wave function after the absorption of a laser photon happens to have a very small overlap with the motional wave function of the excited states. This effect limits the cooling efficiency, since the atoms may remain trapped in these states and not be cooled further, or else cooled much more slowly, toward the ground state. For two ions the probability of finding zeroes of the Franck-Condon coupling is larger than for one ion, as the coupling to the excited state is constituted by two integrals, one for the COM and the other for the relative motion wave functions. Thus the probability of having dark states is higher. To illustrate this phenomenon, in Fig. 5(a) we plot the occupation of the states $P_{\mathbf{n}}$ as a function of the COM and relative vibrational numbers n_0 and n_r, respectively, at a time $t=1000t_F$ after sideband cooling of the COM with $\delta=-2\nu$. Here $\gamma=0.02\nu$, and we are in the limit in which the single resonances are resolved. As a consequence, the most likely coherent transitions are $n_0\rightarrow n_0-2$ and $n_r\rightarrow n_r$. The effect of the dark states is visible in the tail of occupied states of $P_{\mathbf{n}}$, with $n_r=6$ and 7. In Figs. 5(b) and 5(c), we plot the modulus square of the Franck-Condon coefficients for the relative motion corresponding to the coupling of the states $n_r=6$ and 7 to the other motional states: here it is clearly shown that for the transitions $n_r=6\rightarrow 6$ and $n_r=7\rightarrow 7$ the coupling is reduced nearly to zero. As the linewidth γ increases, the number of states to which a single state is coupled increases. Thus the number of channels through which the atom may be cooled is larger. As an effect the dark states disappear. This is shown in Fig. 6, where the population $P_{\mathbf{n}}$ is plotted for $t=600t_F$ and $\gamma=\nu$, and otherwise the same parameters as before. Here we see that the system is cooled homogeneously. The effect of varying γ is summarized in Fig. 7, where we compare the average COM vibrational number vs time in unit t_F for various values of γ. The results of Fig. 7 show clearly that as the linewidth increases the number of fluorescence cycles needed for cooling the system decreases dramatically. It is important to note that in this diagram the time is measured in units of fluorescence cycles for each γ, so that the absolute cooling time clearly reduces more strongly. We stress that this strong dependence on the linewidth is a two-ion effect. In contrast, in sideband cooling of single ions, the fluorescence time t_F determines the cooling time scale for $\gamma/\nu\leqslant 1$, and the curves for different values of γ vs the time in units of the respective t_F do not show striking differences.

The presence of dark states and the coupling of each state to more than one state at almost the same transition frequency might also lead to the formation of *dark coherences* between quasidegenerate states, i.e., to superpositions of states which decouple from laser excitation because of quantum interference. However, for the considered system those dark coherences do not play any significant role. We prove this numerically in Fig. 8, where we plot the comparison between a QMC and a rate equation simulation. We see that there are no striking differences between the two curves. We point out that outside the Lamb-Dicke regime a rate equation treatment is not justified in principle, since secular approximation arguments and Lamb-Dicke limit arguments cannot be applied. Here the rate equations are used to highlight the effect of neglecting the coherences in the dynamical evolution of the cooling, while these coherences are fully accounted for in the QMC treatment. In order to see why coherences do not play any significant role in the cooling dynamics, we look at the definition of a dark coherence. Let us consider a state $|\alpha\rangle$ at $t=0$ defined for simplicity as linear superposition of two quasidegenerate states $|\alpha\rangle=a_1|\mathbf{n}\rangle$

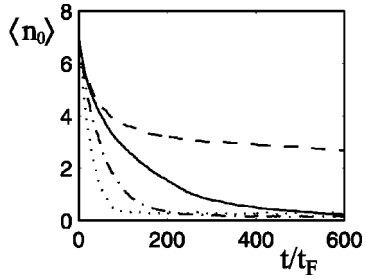

FIG. 7. Time dependence of the average vibrational number of the COM mode for $\gamma=0.02\nu$ (dashed line), $\gamma=0.2\nu$ (solid line), $\gamma=0.4\nu$ (dash-dotted line) and $\gamma=\nu$ (dotted line), keeping constant the ratio $\Omega/\gamma=0.17$. The time is in unit $t_F(\gamma)=2\gamma/\Omega^2\approx 70/\gamma$. All other parameters are the same as in Figs. 4(b) and 4(c).

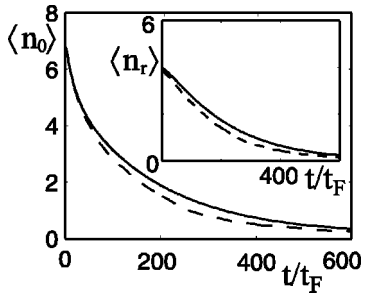

FIG. 8. Comparison between the rate equation (solid line) and QMC calculation (dashed line). Same physical parameters as in Fig. 4(b). The onset refers to the COM, and the inset to the relative motion vibrational number.

$+a_2 e^{i\phi}|\mathbf{m}\rangle$, with a_1, a_2, and ϕ real coefficients and with $\mathbf{n}$ and $\mathbf{m}$ states almost degenerate in energy, so that $|(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\nu}|=\Delta E$ with $\Delta E \leqslant \gamma$. In principle a dark coherence can be a linear superposition of any number of states. However, as we will see from the arguments below, our restriction to two states does not affect the generality of the result. The evolution $|\alpha(t)\rangle$ in the Schrödinger picture, apart from a global phase factor, is written as

$$|\alpha(t)\rangle = a_1|\mathbf{n}\rangle + a_2 e^{i\phi(t)}|\mathbf{m}\rangle, \tag{16}$$

with $\phi(t)=\phi_0+\Delta E t$. The superposition state is dark when the following condition is fulfilled:

$$\langle \mathbf{l}|e^{ik(X+x/2)}|\alpha(t)\rangle \sim 0 \tag{17}$$

for any state $|\mathbf{l}\rangle$ belonging to the set of states $\{|\mathbf{l}\rangle\}$ to which it is resonantly or almost resonantly coupled. If condition (17) holds at $t=0$, it will hold up to a time t such that $\Delta E t \sim \pi/2$. For the system we are dealing with we do not have an exact degeneracy, thus we check whether the state $|\alpha\rangle$ can remain dark for a time sufficiently long to affect the cooling dynamics appreciably. The smallest possible value of ΔE in the range of energies of our calculations is $\Delta E = 0.07\nu$, and we find that ϕ rotates by an angle $\pi/2$ in less than one fluorescence cycle for the values of γ that we have considered. The dark coherences are then washed away during the evolution as an effect of the incommensurate frequencies between the two modes. This result together with the numerical results suggests that rate equations can be used in the study of cooling [24].

In order to highlight that the absence of dark coherences is a signature of the peculiar spectrum of the system, in Fig. 9 we plot the cooling of one mode outside the Lamb-Dicke regime for the case of a discrete spectrum where we have exact degeneracy. More precisely we consider two harmonic oscillators with commensurate frequencies ν and 2ν, where all the other physical parameters are the same as before. In this case the different outcome between the QMC and the rate equation treatment is dramatic, giving evidence of the role of coherences in the evolution.

C. Light on both ions

The calculations that we have shown refer to the case in which only one ion is illuminated. As we have seen, although light interacts with one ion it couples with both

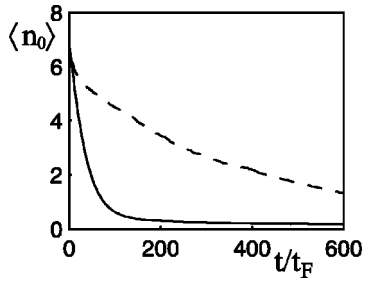

FIG. 9. Plot of the average vibrational number vs time for a harmonic oscillator of frequency ν coupled to a second one with frequency 2ν. Comparison between rate equation (solid line) and QMC calculation (dashed line). The Lamb-Dicke parameter for the mode ν is $\eta_\nu=0.6$. $\gamma=0.2\nu$, $\Omega=0.034\nu$, and $\delta=-2\nu$, and atoms are initially flatly distributed on the states with an energy $E\leqslant 15\nu$.

modes simultaneously, as shown in Eqs. (4) and (13). When both ions are excited by laser light, the system is described by a four-level scheme, corresponding to the four internal states $|a_1,b_2\rangle$ with $a,b=e,g$, where we assume that when a photon is emitted, we detect from which ion the event has occurred, as a consequence of the spatial resolution of the ions. We expect that the effect on the cooling will be a doubling in the number of quantum jumps and hence of the cooling rate. This is shown in Fig. 10, where we compare the time dependence of the COM vibrational number for the cases in which only one ion is illuminated (dashed line) or both ions are illuminated with the same laser intensity (solid line with label 1). In the latter case cooling is visibly faster, and the time dependence scales with a factor of 2 with respect to the case with one ion illuminated, as we can see when we replot the solid line 1 vs $t/2t_F$ (solid line with index 2). The two-ion effects found above are clearly independent of the number of scattering points, with the only difference that the dark coherence condition is now written as $\langle \mathbf{l}|(e^{ik(X+x/2)}+e^{ik(X-x/2)})|\alpha(t)\rangle \sim 0$. We point out again that effects due to interference between the internal excitation paths have been neglected, as we consider the ions to be two distinguishable particles [25].

We would like to stress that in the above calculations we have considered the case of only one ion driven by radiation while the laser wave vector is parallel to the trap axis. However, if one wants to cool two identical ions by shining light

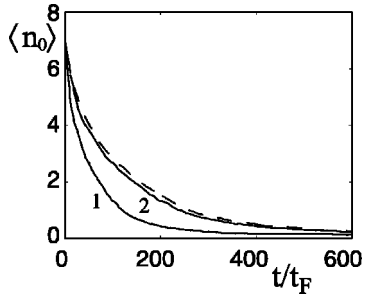

FIG. 10. Plot of the average vibrational number of the COM mode vs the time in unit t_F for the case in which both ions are illuminated (solid line, index 1), and only one ion is illuminated (dashed line). The solid line with index 2 corresponds to line 1 rescaled, where the time has been divided by a factor 2. The other parameters are the same as in Figs. 4(b) and 4(c).

on one of them, the laser beam must necessarily be at a certain angle θ with respect to the ion string. Thus the Lamb-Dicke parameter characterizing the coherent excitation will be smaller than the Lamb-Dicke parameter for the spontaneous decay, and, depending on the minimum angle θ required, the coherent laser will excite with some selectivity one of the two modes. However, for cooling purposes it is preferable to have the two Lamb-Dicke parameters values, corresponding to the spontaneous emission and the coherent excitation, as close as possible.

IV. CONCLUSIONS

In this paper we have studied the question of cooling two ions in a linear trap to the ground state by means of sideband cooling. We have studied sideband cooling in the Lamb-Dicke regime, and have shown that in this limit the two harmonic oscillators can be considered decoupled when one of the two is cooled by means of sideband cooling. We have found that the cooling dynamics in the low-intensity limit may be described by rate equations, and essentially that all the considerations developed for sideband cooling of single ions apply. This regime was considered in Ref. [16] for studying the effects of dipole-dipole interaction in laser cooling of two ions, and by Javanainen in Ref. [17] in his study on laser cooling of ion clusters.

We have then investigated laser cooling outside the Lamb-Dicke regime, finding striking differences with the Lamb-Dicke limit. Here the energy spectrum may be considered a quasicontinuum, though the coupling between the states is still governed by the Franck-Condon coupling. A consequence is that the cooling efficiency depends strongly on the radiative linewidth. For very small ratio γ/ν the effect of dark states is appreciable, and manifests itself in a drastic increase of the cooling time, i.e., the number of fluorescence cycles needed to cool the system. For $\gamma/\nu<1$, but large enough, the effect of dark states is washed away, the modes are cooled simultaneously, and the cooling time is considerably shorter and comparable to the time needed for cooling single ions outside of the Lamb-Dicke regime. These effects are all consequences of the density of states in the energy spectrum. A further property of the system is the absence of dark coherences, as a consequence of the incommensurate frequencies of the harmonic oscillators, i.e., of the absence of perfect degeneracy. This implies that in the low-intensity limit rate equations still provide a good description of the cooling dynamics. Finally, we have compared cooling when light is shone on one ion only or on both ions, finding a simple difference of a factor 2 in the rate of cooling.

On the basis of the obtained results we would like to comment on cooling of a string of $N>2$ ions. N ions in a harmonic trap may be described by N harmonic oscillators. The mode frequencies $\nu_0,\nu_1,\ldots,\nu_{N-1}$ are all incommensurate, and the number of states in the interval of energy $[E,E+\delta E]$ is $D(E)=E^{N-1}\delta E/\nu_0\nu_1\cdots\nu_{N-1}(N-1)!$. Outside the Lamb-Dicke regime the absorption spectrum is then even denser than in the two-ion case, and the probability of dark states will be larger. However, a suitable increase of the linewidth will cancel their effects, since each state will see an even higher number of states, and thus of possible transitions, than for two ions. We expect that this last property will also play a role against the creation of dark coherences. In fact, although on one hand the large density of states may favor the appearance of stable dark coherences between ''accidentally'' degenerate states, on the other hand for values of γ large enough the coherent effects will wash out because of the coupling to a ''continuum'' of states. From these considerations we expect laser cooling to the ground state still to be possible for $N>2$ ions. Furthermore, one can cool a given set of modes to the ground state through the choice of the laser detuning outside the Lamb-Dicke regime. For example, taking a detuning $\delta=-\nu_k$, the modes with frequency $\nu_j \geqslant \nu_k$ may be cooled to their vibrational ground states. It should be noted that as the number of ions N increases, the Lamb-Dicke parameter of each mode decreases approximately as $1/\sqrt{N}$ [19], also allowing one to reach the Lamb-Dicke regime when this condition is not fulfilled for single ions. In this limit the modes may be considered decoupled, and sideband cooling is particularly efficient.

As a last consideration, we note that the behavior of a number of ions $N\geqslant 3$ cooled by light depends on which ions of the string are driven. In fact each position of the string couples with the different modes with amplitudes that depend on the position itself [19]. Only if all the ions are illuminated may we consider all the modes as coupled and cooled simultaneously. But this ''coupling'' changes as we select and drive only certain ions of the string. In this case a certain amount of modes may be cooled, while the others will remain hot or be cooled on a longer time scale.

ACKNOWLEDGMENTS

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APPENDIX A: HARMONIC APPROXIMATION

We consider term (3) and rewrite it in COM and relative motion canonical variables:

$$H_{\rm mec}=\frac{P^2}{2M}+\frac{1}{2}M\nu^2X^2+\frac{p^2}{2\mu}+\frac{1}{2}\mu\nu^2x^2+\frac{e^2}{4\pi\epsilon_0|x|}$$
$$=H(X,P)+H(x,p), \tag{A1}$$

with $\mu=m/2$ the reduced mass, and $M=2m$ the total mass. The mechanical problem is separable into center-of-mass motion and relative motion, where $H(X,P)$ describes the harmonic motion of a particle of mass M interacting with a harmonic oscillator of frequency ν, and $H(x,p)$ the motion of a particle of mass μ interacting with a potential $V(x)=\mu\nu^2x^2/2+e^2/4\pi\epsilon_0|x|$, i.e., a harmonic potential of frequency ν and a central repulsive Coulomb-type potential.

We focus our attention on the potential $V(x)$, restricting its domain on the semiaxis $x>0$. The equilibrium point is found to be $x_0=(e^2/4\pi\epsilon_0\mu\nu^2)^{1/3}$. Expanding $V(x)$ around x_0, we find

$$V(x)=\frac{3}{2}\left(\frac{e^2}{4\pi\epsilon_0}\mu\nu^2\right)^{1/3}+\frac{3}{2}\mu\nu^2(x-x_0)^2+\sum_{n=3}^{\infty}(-1)^n\frac{e^2}{4\pi\epsilon_0 x_0^{n+1}}(x-x_0)^n \tag{A2}$$

$$=\frac{3}{2}\left(\frac{e^2}{4\pi\epsilon_0}\mu\nu^2\right)^{1/3}+\frac{1}{2}\mu\nu_r^2(x-x_0)^2+A(x), \tag{A3}$$

where $\nu_r=\sqrt{3}\nu$ and $A(x)$ sum over the higher order terms, which we call the anharmonic terms. We quantize the oscillation around the equilibrium position x_0,

$$x=x_0+\sqrt{\frac{\hbar}{2m\nu_r}}(a_r^\dagger+a_r), \tag{A4}$$

where a_r and $a_r^\dagger$ are annihilation and creation operators, respectively. From perturbation theory we may consider $V(x)$ harmonic when the following conditions are fulfilled:

$$\langle j|A(x)|j\rangle\ll\hbar\nu_r,$$

$$|\langle j|A(x)|j\pm1\rangle|\ll\hbar\nu_r, \tag{A5}$$

with $|j\rangle$ the eigenstate of the harmonic oscillator of frequency ν_r with eigenvalue $j\hbar\nu_r$. The first condition means that the energy shift due to the anharmonic term is much smaller than the spectrum separation, whereas the second condition means that the coupling between the states is a small perturbation, and we will show that it may be neglected. The coupling between the state j and the state $j+k$ is not taken here into account for simplicity, but it may be shown that it is much smaller than $k\hbar\nu_r$ in a similar way to the one we discuss below. Let us rewrite the relations in Eq. (A5) as

$$\langle j|A(x)|j\rangle=\frac{e^2}{4\pi\epsilon_0 x_0}\sum_{m=2}^{\infty}\zeta_{j,2m},$$

$$\langle j|A(x)|j+1\rangle=-\frac{e^2}{4\pi\epsilon_0 x_0}\sum_{m=1}^{\infty}\zeta_{j,2m+1}, \tag{A6}$$

with

$$\zeta_{j,2m}=\left(\sqrt{\frac{\hbar}{x_0^2 2\mu\nu_r}}\right)^{2m}\langle j|(a_r^\dagger+a_r)^{2m}|j\rangle, \tag{A7}$$

where $\zeta_{j,2m+1}$ is analogously defined. From the relations

$$\frac{(j+m)!}{j!}<\langle j|(a_r^\dagger+a_r)^{2m}|j\rangle<\frac{(2m)!}{m!m!}\frac{(j+m)!}{j!}<2^{2m}\frac{(j+m)!}{j!},$$

$$\frac{(j+m)!}{j!}\sqrt{j+m+1}<\langle j|(a_r^\dagger+a_r)^{2m+1}|j\pm1\rangle<\frac{(2m+1)!}{m!(m+1)!}\frac{(j+m)!}{j!}\sqrt{j+m+1}<2^{2m+1}\frac{(j+m)!}{j!}\sqrt{j+m+1}, \tag{A8}$$

we find that

$$\frac{|\zeta_{j,2m+1}|}{|\zeta_{j,2m}|}\sim\left(2\sqrt{\frac{\hbar}{x_0^2 2\mu\nu_r}}\right)^{2m}\sqrt{j+m+1}. \tag{A9}$$

From Eq. (A9) we see that the series does not converge, as the term of the expansion depends on the term m. However for a certain interval corresponding to $j,m\leqslant M_0$ such that $\psi=\sqrt{\hbar/x_0^2 2\mu\nu_r}\sqrt{M_0}\ll1$, each term is bounded by the corresponding term of a geometrical series with factor $\psi\ll1$. For typical values of a linear ion trap [12], $\psi\sim\sqrt{M_0}/100$, so that M_0 can assume very large values, $M_0\ll10^4$.

The divergence at orders $m>M_0$ is a signature of the divergence of the Coulomb potential at $x=0$: such divergence requires the wave functions to be zero in $x\leqslant0$, whereas the harmonic-oscillator wave functions are different from zero on all the space, although their occupation on the semiaxis $x\leqslant0$ is very small for $\psi\ll1$. The divergence at $x=0$ is a mathematical consequence, and does not correspond to the physical situation, in which the ions move in a three-dimensional space, although the confinement in the other two dimensions is relatively tight. In principle one could build up a potential that does not diverge in a specific point of the x axis, whose behavior for $x>0$ tends to the Coulomb one. In this limit the series should converge for low number states j. However, such a detailed study is beyond the scope of this paper. A discussion can be found in Ref. [20]. We restrict ourselves to the case in which $\psi\ll1$, showing that in the chosen range the harmonic approximation is quite good.

From this consideration it is then sufficient to compare the third-order term with the harmonic potential in order to show that the approximation is sensible. The relation to be satisfied is then

$$\hbar\nu_r\gg\frac{e^2}{4\pi\epsilon_0 x_0^4}\left(\sqrt{\frac{\hbar}{2\mu\nu_r}}\right)^3 j^{3/2}, \tag{A10}$$

with $j\leqslant M_0$, which poses a further condition on j and M_0. Manipulating the expression we find

$$j\ll j_{\max}\approx\left(\frac{\hbar\nu_r}{e^2/4\pi\epsilon_0 x_0}\right)^{2/3}\left(\frac{x_0}{\sqrt{\hbar/2\mu\nu_r}}\right)^2, \tag{A11}$$

which substantially agrees with the qualitative estimate in Ref. [26]. For $j \ll j_{\max}$ the potential can be considered harmonic. For linear traps [12], $j_{\max} \sim 120$, and the harmonic approximation is valid for the region of energy we are considering. As an example the case $j=100$ corresponds to a correction of the order of $10^{-3}\hbar\nu_r$.

APPENDIX B: ADIABATIC ELIMINATION OF THE EXCITED STATE

We rewrite Eq. (7) in the following way:

$$\frac{d}{dt}\rho = -\frac{i}{\hbar}[H_{\rm eff}\rho - \rho H_{\rm eff}^{\dagger}] - \frac{i}{\hbar}[V,\rho] + J\rho, \tag{B1}$$

where $H_{\rm eff}$ is the effective Hamiltonian

$$H_{\rm eff} = H_i + H_{\rm mec} - i\frac{\gamma}{2}\sum_{i=1,2}\sigma_i^{\dagger}\sigma_i, \tag{B2}$$

and $J\rho$ the jump operator

$$J\rho = \gamma\sum_{i=1,2}\sigma_i\tilde{\rho}\sigma_i^{\dagger}. \tag{B3}$$

We introduce the Liouvillians

$$L_0\rho = -\frac{i}{\hbar}[H_{\rm eff}\rho - \rho H_{\rm eff}^{\dagger}], \tag{B4}$$

$$L_1\rho = -\frac{i}{\hbar}[V,\rho],$$

$$L_2\rho = J\rho, \tag{B5}$$

so that Eq. (B1) can be rewritten as

$$\frac{d}{dt}\rho = [L_0 + L_1 + L_2]\rho. \tag{B6}$$

In the limit $\Omega \ll \gamma$ we can eliminate the excited state in second-order perturbation theory. Calling P the projector onto the internal ground state $|g\rangle$, and using a standard derivation based on projectors [27], we obtain the following equation for the ground state of the system:

$$\frac{d}{dt}P\rho(t) = PL_0P\rho(t) + \int_0^t d\tau\, PL_1(1-P)\exp(L_0\tau)(1-P)L_1P\rho(t-\tau) + \int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 PL_2(1-P)\exp(L_0\tau_1)(1-P)L_1\exp(L_0(\tau_2-\tau_1))P\rho(t-\tau_2), \tag{B7}$$

where P is a projector so defined on a density operator X, $PX = |g\rangle\langle g|X|g\rangle\langle g|$. The Markov approximation can be applied in the limit in which we may consider the coupling to the excited state to evolve at a higher rate respect to the time scale which characterizes the ground-state evolution. This is true once we have moved to the interaction picture with respect to the trap frequency. We define

$$v_I(t) = e^{iH_{\rm mec}t}P\rho(t)e^{-iH_{\rm mec}t}, \tag{B8}$$

and in the interaction picture Eq. (B7) has the form

$$\frac{d}{dt}v_I(t) = e^{iH_{\rm mec}t}\left[\int_0^t d\tau\, PL_1\exp(L_0\tau)L_1e^{-iH_{\rm mec}(t-\tau)}v_I(t-\tau)e^{iH_{\rm mec}(t-\tau)}\right]e^{-iH_{\rm mec}t} + e^{iH_{\rm mec}t}\left[\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 PL_2\exp(L_0\tau_1)L_1\exp(L_0(\tau_2-\tau_1))L_1e^{-iH_{\rm mec}(t-\tau_2)}v_I(t-\tau_2)e^{iH_{\rm mec}(t-\tau_2)}\right]e^{-iH_{\rm mec}t}. \tag{B9}$$

Now we can neglect the change of v_I during the time τ on which the excited state evolves. Going back to the original reference frame, we have now the equation in the Markov approximation:

$$\frac{d}{dt}P\rho(t) = \int_0^{\infty} d\tau\, PL_1\exp(L_0\tau)L_1e^{iH_0\tau}P\rho(t)e^{-iH_0\tau} + \int_0^{\infty} d\tau_1\int_{\tau_1}^{\infty} d\tau_2 PL_2\exp(L_0\tau_1)L_1\exp(L_0(\tau_2-\tau_1))L_1e^{iH_0\tau_2}P\rho(t)e^{-iH_0\tau_2}. \tag{B10}$$

We substitute now into Eq. (B10) the explicit form of the operators. After some algebra and application of the commutation rules, we obtain the following equation (where we have neglected the interference terms between the two ions):

$$\frac{d}{dt}P\rho(t)=PL_0\rho(t)-\sum_{i=1,2}\frac{\Omega_i^2}{4}P\int_0^\infty d\tau[e^{(i\delta-\gamma/2)\tau}e^{ikx_i}e^{-iH_{\rm mec}\tau}e^{-ikx_i}e^{iH_{\rm mec}\tau}P\rho_1(t)$$

$$+e^{-(i\delta+\gamma/2)\tau}P\rho(t)e^{-iH_{\rm mec}\tau}e^{ikx_i}e^{iH_{\rm mec}\tau}e^{-ikx_i}]$$

$$+\sum_{i=1,2}\frac{\Omega_i^2}{4}P\int_0^\infty d\tau_1\int_{\tau_1}^\infty d\tau_2\int_{-1}^1 du\,N(u)e^{ikux_i}$$

$$\times[e^{i\delta(\tau_1-\tau_2)-\gamma/2(\tau_1+\tau_2)}e^{-iH_{\rm mec}\tau_1}e^{-ikx_i}e^{iH_{\rm mec}\tau_1}P\rho(t)e^{-iH_{\rm mec}\tau_2}e^{ikx_i}e^{iH_{\rm mec}\tau_2}$$

$$+e^{-i\delta(\tau_1-\tau_2)-\gamma/2(\tau_1+\tau_2)}e^{-iH_{\rm mec}\tau_2}e^{-ikx_i}e^{iH_{\rm mec}\tau_2}P\rho(t)e^{-iH_{\rm mec}\tau_1}e^{ikx_i}e^{iH_{\rm mec}\tau_1}]e^{-ikux_i}. \quad \text{(B11)}$$

Projecting Eq. (B11) on the basis of states $\{|\mathbf{n}\rangle\}$, we obtain Eq. (10).

[1] An overview of laser cooling can be found in S. Chu, Rev. Mod. Phys. **70**, 685 (1998); C. Cohen-Tannoudij, *ibid.* **70**, 707 (1998); W. D. Phillips, *ibid.* **70**, 721 (1998).

[2] H. Perrin, A. Kuhn, I. Bouchoule, and C. Salomon, Europhys. Lett. **42**, 395 (1998); S. E. Hamann, D.L. Haycock, G. Klose, P. H. Pax, I. H. Deutsch, and P. S. Jessen, Phys. Rev. Lett. **80**, 4149 (1998).

[3] A theoretical study about possibilities of achieving Bose-Einstein condensation with laser cooling of neutral atoms can be found in Y. Castin, J. I. Cirac, and M. Lewenstein, Phys. Rev. Lett. **80**, 5305 (1998), and references therein.

[4] A discussion about the advantages of using laser cooling for achieving a quantum statistical regime can be found in J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. A **51**, 2899 (1995), and references therein.

[5] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995). For a review on the ion trap quantum computer, see A. Steane, Appl. Phys. B: Photophys. Laser Chem. **64**, 623 (1997).

[6] H. C. Naegerl, D. Leibfried, H. Rohde, G. Thalhammer, J. Eschner, F. Schmidt-Kaler, and R. Blatt, Phys. Rev. A (to be published).

[7] Recently a scheme of addressing two ions individually has been demonstrated, where the ions are not optically resolved but instead their micromotion is used; see Q. A. Turchette, C. S. Wood, B. E. King, C. J. Myatt, D. Leibfried, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. Lett. **81**, 3631 (1998). A generalization of this technique for more than two ions has also been proposed; see D. Leibfried, Los Alamos archives, e-print quant-ph/9812033.

[8] S. Stenholm, Rev. Mod. Phys. **58**, 699 (1986).

[9] F. Diedrich, J. C. Berquist, W. M. Itano, and D. J. Wineland, Phys. Rev. Lett. **62**, 403 (1989); C. Monroe, D. M. Meekhof, B. E. King, S. R. Jefferts, W. M. Itano, D. J. Wineland, and P. Gould, *ibid.* **75**, 4011 (1995).

[10] I. Marzoli, J. I. Cirac, R. Blatt, and P. Zoller, Phys. Rev. A **49**, 2771 (1994).

[11] B. E. King, C. S. Wood, C. J. Myatt, Q. A. Turchette, D. Leibfried, W. M. Itano, C. Monroe, and D. J. Wineland, Phys. Rev. Lett. **81**, 1525 (1998).

[12] H. C. Naegerl, W. Bechter, J. Eschner, F. Schmidt-Kaler, and R. Blatt, Appl. Phys. B: Photophys. Laser Chem. **66**, 603 (1998); R. J. Hughes, D. F. V. James, J. J. Gomez, M. S. Gulley, M. H. Holzscheiter, P. G. Kwiat, S. K. Lamoreaux, C. G. Peterson, V. D. Sandberg, M. M. Schauer, C. M. Simmons, C. E. Thorbum, D. Tupa, P. Z. Wang, and A. G. White, Fortschr. Phys. **46**, 329 (1998).

[13] M. G. Raizen, J. M. Gilligan, J. C. Bergquist, W. M. Itano, and D. J. Wineland, Phys. Rev. A **45**, 6493 (1992); M. Drewsen, C. Brodersen, L. Hornekaer, J. S. Hangst, and J. P. Schiffer, Phys. Rev. Lett. **81**, 2878 (1998); W. Alt, M. Block, P. Seibert, and G. Werth, Phys. Rev. A **58**, R23 (1998); I. Waki, S. Kassner, G. Birkl, and H. Walther, Phys. Rev. Lett. **68**, 2007 (1992).

[14] G. Morigi, J. I. Cirac, M. Lewenstein, and P. Zoller, Europhys. Lett. **39**, 13 (1997).

[15] G. Morigi, J. I. Cirac, K. Ellinger, and P. Zoller, Phys. Rev. A **57**, 2909 (1998); D. Stevens, A. Brochards, and A. M. Steane, Phys. Rev. A **58**, 2750 (1998); L. Santos and M. Lewenstein, preprint quant-ph/9808014.

[16] A. W. Vogt, J. I. Cirac, and P. Zoller, Phys. Rev. A **53**, 950 (1996).

[17] J. Javanainen, J. Opt. Soc. Am. B **5**, 73 (1988).

[18] J. N. Tan, J. J. Bollinger, B. Jelenkovic, and D. J. Wineland, Phys. Rev. Lett. **75**, 4198 (1995); F. Diedrich, E. Peik, J. M. Chen, W. Quint, and H. Walther, *ibid.* **59**, 2935 (1987); W. M. Itano, J. J. Bollinger, J. N. Tan, B. Jelenkovic, X.-P. Huang, and D. J. Wineland, Science **279**, 686 (1998); G. Birkl, S. Kassner, and H. Walther, Nature (London) **357**, 310 (1992).

[19] D. F. V. James, Appl. Phys. B: Photophys. Laser Chem. **66**, 181 (1998).

[20] J. Yin and J. Javanainen, Phys. Rev. A **51**, 3959 (1995).

[21] W. Lange (private communication).

[22] The final energy obtained by means of Doppler cooling in one dimension of single atoms in a trap depends on the internal transition quantum numbers, and for hyperfine states with selection rule $\Delta m_F=\pm 1$ it is $\langle n\rangle\nu=7\gamma/20$; see Ref. [8].

[23] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).

[24] This argument is valid in the limit of low saturation. At saturation the antibunched nature of the atoms' spontaneous emission plays a role by modifying the diffusion. See Ref. [8].

[25] Phenomena due to interference between the internal paths of excitation, which lead to superradiance and subradiance effects, occur for distances between the ions of the order of the

wavelength, when dipole-dipole interaction cannot be neglected. See R. G. DeVoe and R. G. Brewer, Phys. Rev. Lett. **76**, 2049 (1996); R. G. Brewer, *ibid.* **77**, 5153 (1996); and Ref. [16].

[26] V. V. Dodonov, V. I. Manko, and L. Rosa, Phys. Rev. A **57**, 2851 (1998).

[27] C. W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991).