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The exciton many-body theory extended to any kind of composite bosons

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Abstract. We have recently constructed a many-body theory for composite excitons, in which the possible carrier exchanges between N excitons can be treated exactly through a set of dimensionless "Pauli scatterings" between two excitons. Many-body effects with free excitons turn out to be rather simple because these excitons are the exact one-pair eigenstates of the semiconductor Hamiltonian, in the absence of localized traps. They consequently form a complete orthogonal basis for one-pair states. As essentially all quantum particles known as bosons are composite bosons, it is highly desirable to extend this free exciton many-body theory to other kinds of "cobosons" — a contraction for composite bosons — the physically relevant ones being possibly not the exact one-pair eigenstates of the system Hamiltonian. The purpose of this paper is to derive the "Pauli scatterings" and the "interaction scatterings" of these cobosons in terms of their wave functions and the interactions which exist between the fermions from which they are constructed. It is also explained how to calculate many-body effects in such a very general composite boson system.

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A few years ago, we have tackled the difficult problem of many-body effects between composite bosons through the study of interacting excitons in semiconductors [1–3]. Excitons actually constitute a very nice "toy model" to study the consequences of the boson composite nature, since the semiconductor Hamiltonian is extremely simple — just electrons and holes with kinetic energy and Coulomb interaction — the full spectrum of bound and unbound exciton eigenstates being analytically known in terms of hypergeometric functions, in 3D and 2D. When we started these studies, we had in mind to better understand the bosonization procedures [4] and to properly determine their limit of validity, through full-proof ab initio calculations. To our major surprise — and contentment — we have found that, whatever the effective Hamiltonians for bosonized excitons [5] are, they miss a set of processes which actually produce the dominant terms in various problems of physical interest, such as the semiconductor optical nonlinearities.

The many-body theory we have constructed, which only uses the semiconductor Hamiltonian written in terms of free electrons and free holes, makes appearing *two* fully independent scatterings [1–3]: one is associated to direct Coulomb processes between two excitons, the "in" and "out" excitons being made with the same electron-hole pairs. The second scattering is completely novel. It directly comes from the undistinguishability of the fermionic components of the excitons, and describes the carrier exchanges which can take place between two excitons, in the absence of any Coulomb process. While the direct Coulomb scatterings ξ are energy-like quantities, these novel "Pauli scatterings" λ are dimensionless, so that they are, by construction, missed in any model Hamiltonian for interacting excitons, *whatever* the effective scatterings of these Hamiltonians are — a very strong statement, indeed!

Using dimensional arguments only, it is possible to show [6–8] that the semiconductor optical nonlinearities are entirely controlled by these Pauli scatterings at large detuning, so that there is no hope to correctly describe these nonlinearities through effective Hamiltonians for bosonized excitons.

We could hope a better correctness in physical effects controlled by energy-like scatterings, such as the scattering rates of two excitons. Unfortunately, this is not true: we have shown [9] that, in order to recover the correct value of these quantities, the effective scatterings between excitons that must be introduced in the exciton Hamiltonian, make this Hamiltonian non hermitian — although

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not exactly the same as the usual exciton Hamiltonian — a major physical failure hard to accept. Moreover, the links between the lifetime and the sum of scattering rates for elementary and composite bosons differ by a factor of 2, so that there is no way to find effective scatterings between bosonized excitons giving the lifetime and the scattering rates correctly.

All this led us to conclude that, in order to correctly describe many-body effects with excitons, it is not possible to "cook" the Coulomb interactions between electrons and holes with carrier exchanges, once and for all, in a set of "Coulomb scatterings dressed by exchange" as done in all model Hamiltonians describing interacting excitons.

Since essentially all quantum particles known as bosons are composite particles, similar problems are expected to appear in their many-body effects. This is why it is highly desirable to extend this new many-body theory for excitons to any type of composite bosons, i.e., to formally write their Pauli and interaction scatterings without using any particular form for these composite bosons nor for the system Hamiltonian.

The present paper is organized as follows.

In the first section, we settle the notations and define the composite bosons we study in this work, through their expansion in terms of free fermions α and free fermions β. In order to possibly describe any system of fermion pairs in terms of cobosons, it is necessary to consider that these cobosons form a complete set for one-fermion-pair states. However, this does not impose the one-coboson states to be orthogonal: We already had to deal with nonorthogonal cobosons in one of our recent works on electron teleportation between quantum dots induced by unabsorbed laser pulses [10], one of the composite bosons of physical interest being a pair of non interacting trapped electrons.

In Section 2, we determine the Pauli scatterings due to fermion exchanges between these composite bosons. As physically reasonable, they only depend on the composite bosons of interest, through their wave functions, but not on the system Hamiltonian. We, in particular, show how our results on the scalar products of exciton-states can be readily extended to arbitrary composite bosons, even if the one-coboson states are not orthogonal.

In Section 3, we show how we can formally write the energy-like interaction scatterings for any type of composite bosons — not necessarily the eigenstates of the system Hamiltonian — in terms of the potentials between fermions α and β appearing in this Hamiltonian.

In a last section, we explain how to derive many-body effects between these composite bosons, following a path similar to the one we have used for excitons.

As our works on exciton many-body effects have quite clearly pointed out the many weaknesses of the bosonization procedures, while the many-body theory we have constructed, now allows to treat the fermion exchanges between composite particles exactly, it can be of interest to introduce a new name for these composite quantum particles, the "coboson" — a contraction of "composite boson" — since they now have their specific many-body theory as well as their specific "Shiva diagrams", more elaborated than the Feynman diagrams for elementary bosons, due to the various fermion exchanges which take place between composite quantum particles.

1 The most general composite bosons

Let us consider a quantum particle made of one fermion α and one fermion β. It is convenient to introduce two orthogonal basis for these fermions, these basis being a priori arbitrary,

$$|\mathbf{k}_\alpha\rangle = a^\dagger_{\mathbf{k}_\alpha}|v\rangle$$
$$|\mathbf{k}_\beta\rangle = b^\dagger_{\mathbf{k}_\beta}|v\rangle. \qquad (1)$$

The anticommutators of their creation operators are such that $\{a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\}_+ = \delta_{\mathbf{k}',\mathbf{k}} = \{b_{\mathbf{k}'}, b^\dagger_{\mathbf{k}}\}_+$.

The states $|\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle = a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta}|v\rangle$ form a complete set for one fermion pair (α,β), so that the closure relation for one-pair states reads

$$I = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} |\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle\,\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|. \qquad (2)$$

This closure relation allows to write any state $|i\rangle$ made of one (α,β) pair as

$$|i\rangle = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} |\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle\,\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle. \qquad (3)$$

By writing this one-pair state $|i\rangle$ as $B^\dagger_i|v\rangle$, we readily deduce that the creation operator $B^\dagger_i$ reads in terms of creation operators for free fermions α and β, as

$$B^\dagger_i = \sum_{\mathbf{k}_\alpha,\mathbf{k}_\beta} a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta}\,\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle. \qquad (4)$$

Being made of a pair of fermion operators, $B^\dagger_i$ is a composite boson creation operator, whatever $\langle\mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle$ is.

In order to possibly describe any system of (α,β) pairs entirely in terms of cobosons, it is necessary for these cobosons to form a complete set for one-pair states. If the coboson states $|i\rangle$ are normalized and orthogonal, as for the $|i\rangle$'s being Hamiltonian eigenstates, their closure relation simply reads

$$I = \sum_i |i\rangle\,\langle i|. \qquad (5)$$

This allows to write the creation operator for a free fermion pair in terms of coboson creation operators as

$$a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta} = \sum_i B^\dagger_i\,\langle i|\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle. \qquad (6)$$

If the physically relevant cobosons form a complete set, but if this set is not orthogonal — as the pairs of trapped electrons we have studied in reference [10], — their closure relation is not as simple as equation (5). It now reads

$$I = \sum_{i,j} |i\rangle\, z_{ij}\,\langle j|, \qquad (7)$$

where the prefactors z_{ij} are such that

$$\sum_m z_{im}\,\langle m|j\rangle = \delta_{ij}. \tag{8}$$

The above equation just says that the matrix made of the z_{ij}'s and the matrix made of the $\langle i|j\rangle$'s are inverse matrices. For nonorthogonal cobosons, the link between free pair and coboson creation operators now reads, instead of equation (6),

$$a^\dagger_{\mathbf{k}_\alpha} b^\dagger_{\mathbf{k}_\beta} = \sum_{i,j} B^\dagger_i\, z_{ij}\,\langle j|\mathbf{k}_\alpha,\mathbf{k}_\beta\rangle. \tag{9}$$

2 Coboson scattering due to fermion exchange

The "interactions" between two composite bosons which only come from the fact that these cobosons can exchange their fermions, do not depend on the forces acting on these fermions. Consequently, to determine these "Pauli scatterings", it is not necessary to specify the system Hamiltonian at hand.

2.1 "Deviation-from-boson operator"

By using equation (4) for the coboson creation operators, we readily get from equation (2),

$$[B_m, B^\dagger_i] = \langle m|i\rangle - D_{mi}, \tag{10}$$

where D_{mi} is the "deviation-from-boson operator". This operator, which is such that $D_{mi}|v\rangle = 0$, as obtained by multiplying the above equation by $|v\rangle$ on the right, in fact appears as $D^{(\alpha)}_{mi} + D^{(\beta)}_{mi}$. In the $D^{(\alpha)}_{mi}$ part, given by

$$D^{(\alpha)}_{mi} = \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta} b^\dagger_{\mathbf{k}_\beta} b_{\mathbf{k}'_\beta} \sum_{\mathbf{k}_\alpha}\langle m|\mathbf{k}_\alpha,\mathbf{k}'_\beta\rangle\,\langle \mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle, \tag{11}$$

the cobosons m and i are made with the same fermion α, while in $D^{(\beta)}_{mi}$, given by

$$D^{(\beta)}_{mi} = \sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha} a^\dagger_{\mathbf{k}_\alpha} a_{\mathbf{k}'_\alpha} \sum_{\mathbf{k}_\beta}\langle m|\mathbf{k}'_\alpha,\mathbf{k}_\beta\rangle\,\langle \mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle, \tag{12}$$

the cobosons m and i are made with the same fermion β.

2.2 "Pauli scatterings" for cobosons

To go further and deduce the "Pauli scatterings" between cobosons, it is necessary to consider that these cobosons form a complete basis for one-pair states, in order to possibly write a pair of free fermions (α,β) in terms of cobosons, using equations (6) or (9).

2.2.1 Orthogonal cobosons

Let us start with orthogonal cobosons related to free pairs through equation (6). The "Pauli scatterings" $\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right)$, due to fermion exchanges between composite particles, are defined through

$$\left[D_{mi}, B^\dagger_j\right] = \sum_n \left[\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right) + \lambda\left(\begin{smallmatrix} m & j\\ n & i\end{smallmatrix}\right)\right] B^\dagger_n. \tag{13}$$

To calculate λ, we use equations (4, 11) to get

$$\left[D^{(\alpha)}_{mi}, B^\dagger_j\right] = \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta,\mathbf{k}_\alpha}\ \sum_{\mathbf{p}_\alpha,\mathbf{p}_\beta} \langle m|\mathbf{k}_\alpha,\mathbf{k}'_\beta\rangle \times \langle \mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle\,\langle \mathbf{p}_\beta,\mathbf{p}_\alpha|j\rangle\,\delta_{\mathbf{k}'_\beta,\mathbf{p}_\beta}\, a^\dagger_{\mathbf{p}_\alpha} b^\dagger_{\mathbf{k}_\beta}. \tag{14}$$

We then express $a^\dagger b^\dagger$ in terms of $B^\dagger$ according to equation (6). This leads to

$$\left[D^{(\alpha)}_{mi}, B^\dagger_j\right] = \sum_n \lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right) B^\dagger_n, \tag{15}$$

where $\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right)$ is given by

$$\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right) = \sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha,\mathbf{k}'_\beta,\mathbf{k}_\beta} \langle m|\mathbf{k}_\alpha,\mathbf{k}'_\beta\rangle\,\langle n|\mathbf{k}'_\alpha,\mathbf{k}_\beta\rangle\,\langle \mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle\,\langle \mathbf{k}'_\beta,\mathbf{k}'_\alpha|j\rangle. \tag{16}$$

The second term on the RHS of equation (13) is obtained in the same way, by calculating $\left[D^{(\beta)}_{mi}, B^\dagger_j\right]$.

The Pauli scattering $\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right)$ is shown in Figure 1a. As for composite excitons, it corresponds to a fermion exchange between the "in" cobosons (i,j) such that the coboson m ends by having the same fermion α as the coboson i. (By convention, the cobosons of the lowest line of the Pauli scattering $\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right)$, here m and i, are made with the same fermion α).

We can rewrite this Pauli scattering in $\mathbf{r}$ space by using

$$\langle \mathbf{k}_\beta,\mathbf{k}_\alpha|i\rangle = \int d\mathbf{r}_\alpha\, d\mathbf{r}_\beta\,\langle \mathbf{k}_\beta|\mathbf{r}_\beta\rangle\,\langle \mathbf{k}_\alpha|\mathbf{r}_\alpha\rangle\,\langle \mathbf{r}_\beta,\mathbf{r}_\alpha|i\rangle, \tag{17}$$

and by performing the summation over the various $\mathbf{k}$'s through closure relations. We find that $\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right)$ reads in terms of the wave functions of the (m,n) and (i,j) cobosons, as

$$\lambda\left(\begin{smallmatrix} n & j\\ m & i\end{smallmatrix}\right) = \int d\mathbf{r}_{\alpha_1}\, d\mathbf{r}_{\alpha_2}\, d\mathbf{r}_{\beta_1}\, d\mathbf{r}_{\beta_2}\, \phi^*_m(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_2}) \times \phi^*_n(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_1})\,\phi_i(\mathbf{r}_{\alpha_1},\mathbf{r}_{\beta_1})\,\phi_j(\mathbf{r}_{\alpha_2},\mathbf{r}_{\beta_2}), \tag{18}$$

where $\phi_i(\mathbf{r}_\alpha,\mathbf{r}_\beta) = \langle \mathbf{r}_\beta,\mathbf{r}_\alpha|i\rangle$ is the wave function of the coboson i (see Fig. 2a).

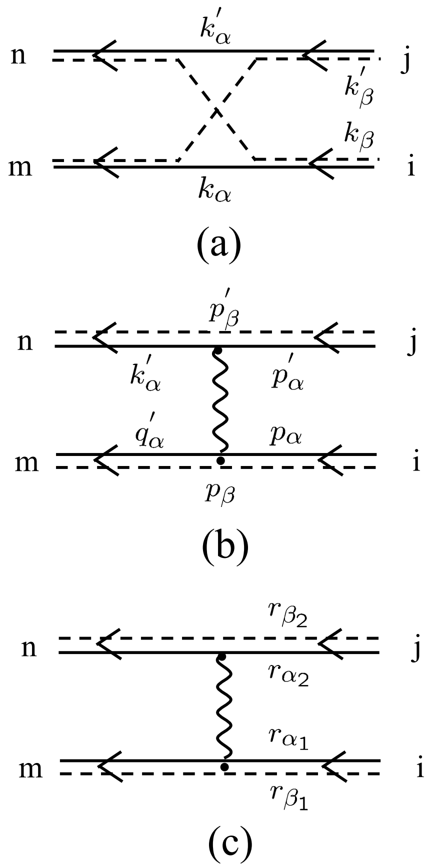

Fig. 1. (a): Diagram, in the free fermion basis $(\mathbf{k}_\alpha, \mathbf{k}_\beta)$, for the "Pauli scattering" $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$ given in equation (16), in which the "in" composite bosons i and j exchange their fermions β, represented by a dashed line, the "out" coboson m being made with the same fermion α as the coboson i. (b): Diagram, in the free fermion basis, of the part $\xi_1\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, given in equation (49), of the interaction scattering, due to interactions between the fermions α of the "in" cobosons (i, j), the "out" cobosons (m, n) being made with the same pairs as the "in" cobosons. (c): Same $\xi_1\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, due to (α, α) interactions, as shown in (b), but now in real space.

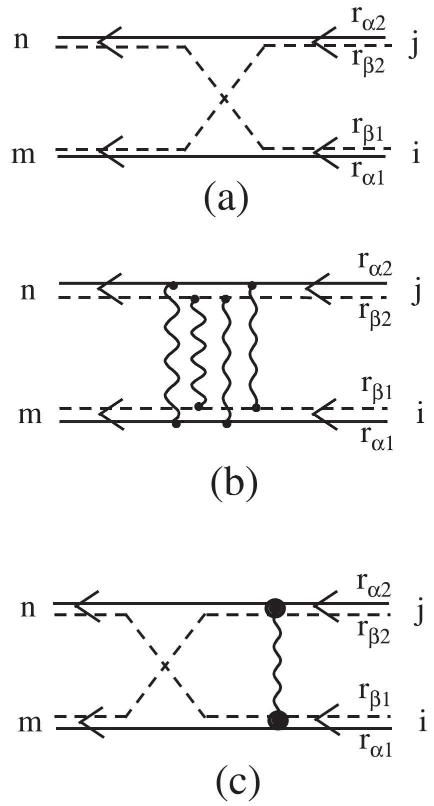

Fig. 2. (a): Diagram for the "Pauli scattering" $\lambda\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, shown in Figure 1a, but now in real space. (b): Diagram, in real space, for the "interaction scattering" $\xi\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, defined in equation (51), in which the "in" cobosons interact through the interactions of the fermions from which they are constructed, the "in" and "out" cobosons being made with the same fermion pairs (α_1, β_1) and (α_2, β_2). (c): Diagram, in real space, for $\xi^{\text{in}}\left(\begin{smallmatrix} n & j \\ m & i \end{smallmatrix}\right)$, defined in equation (54), which is a mixed exchange-interaction scattering, the interactions taking place before the fermion exchange, i.e., between the "in" cobosons (i, j).

2.2.2 Nonorthogonal cobosons

If the cobosons form a nonorthogonal basis for one-pair states, the link between the creation operators for free fermion pairs and cobosons given in equation (6) has to be replaced by the link given in equation (9). From it, we now get

$$\left[D_{mi}, B_j^\dagger\right] = \sum_n B_n^\dagger \sum_p z_{np} \left[\lambda\begin{pmatrix} p & j \\ m & i \end{pmatrix} + \lambda\begin{pmatrix} m & j \\ p & i \end{pmatrix}\right], \quad (19)$$

where $\lambda\begin{pmatrix} p & j \\ m & i \end{pmatrix}$ is the same Pauli scattering as the one defined in equations (16) or (18).

2.3 Scalar product of coboson states

Equation (19) for cobosons forming a nonorthogonal basis is definitely not as simple as equation (13). This, however, has no major consequence on the scalar product of N-coboson states. Indeed, if we consider the scalar product of two coboson states, we find, using equation (10),

$$\langle v|B_m B_n B_i^\dagger B_j^\dagger|v\rangle = \langle n|i\rangle\,\langle m|j\rangle + \langle n|j\rangle\,\langle m|i\rangle - \langle v|B_m D_{ni} B_j^\dagger|v\rangle, \quad (20)$$

the last term of the above equation reading, due to equation (19),

$$\langle v|B_m D_{ni} B_j^\dagger|v\rangle = \sum_{p,q} \langle m|p\rangle\, z_{pq} \left[\lambda\begin{pmatrix} q & j \\ n & i \end{pmatrix} + \lambda\begin{pmatrix} n & j \\ q & i \end{pmatrix}\right]. \quad (21)$$

So that, due to equation (8), the scalar product of two-coboson states reduces to

$$\langle v|B_m B_n B_i^\dagger B_j^\dagger|v\rangle = \langle n|i\rangle\,\langle m|j\rangle + \langle n|j\rangle\,\langle m|i\rangle - \lambda\begin{pmatrix} n & j \\ m & i \end{pmatrix} - \lambda\begin{pmatrix} m & j \\ n & i \end{pmatrix}. \quad (22)$$

The exchange part of this scalar product is just the one for orthogonal cobosons — or for excitons [1] —, the only difference coming from the naïve part, i.e., the part which remains when cobosons are replaced by elementary particles, the scalar product $\langle m|i\rangle$ being just replaced by $\delta_{m,i}$ if the cobosons are orthogonal.

It is possible to show that this nicely simple result can be extended to more complicated scalar products of coboson states.

3 Coboson scatterings due to interactions between fermions

The cobosons interact through fermion exchanges as described in the preceding section. They also interact, in a more standard way, through the forces which exist between the fermions from which they are constructed. It is of importance to note that this second coboson interaction, which can appear as rather naïve, is in fact very subtle due to the fermion undistinguishability. Indeed, with fermions $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, two kinds of cobosons can be made, (α_1, β_1) and (α_2, β_2), or (α_1, β_2) and (α_2, β_1). Due to this, the interactions *between* cobosons resulting from forces between fermions α and β, must be taken as $v(\alpha_1, \beta_2) + v(\alpha_2, \beta_1)$ in the first case, but $v(\alpha_1, \beta_1) + v(\alpha_2, \beta_2)$ in the second case. Since there is no way to know with which pairs of fermions the cobosons are made, there is no way to unambiguously write the interactions *between* cobosons associated to the forces between fermions α and β.

It is however clear that, even if the interactions between cobosons due to forces between fermions α and β cannot be properly defined, these forces must play a role in the many-body physics of these cobosons. The clean way to make them appearing is actually non standard. It again relies on a set of commutators.

3.1 System Hamiltonian for fermions α and β

The general form for a system Hamiltonian made of fermions α and β reads in first quantization as

$$H = H_\alpha + H_\beta + V_{\alpha\alpha} + V_{\beta\beta} + V_{\alpha\beta}. \tag{23}$$

H_α and H_β are one-body operators for fermions α and fermions β:

$$H_\alpha = \sum_n h_\alpha(\mathbf{r}_{\alpha n}). \qquad H_\beta = \sum_n h_\beta(\mathbf{r}_{\beta n}). \tag{24}$$

The three other terms of the Hamiltonian (23) are two-body operators which correspond to interactions between cobosons α, between cobosons β and between cobosons α and β:

$$V_{\alpha\alpha} = \frac{1}{2}\sum_{n\neq n'} v_{\alpha\alpha}(\mathbf{r}_{\alpha n}, \mathbf{r}_{\alpha n'}) \tag{25}$$

$$V_{\alpha\beta} = \sum_{n,n'} v_{\alpha\beta}(\mathbf{r}_{\alpha n}, \mathbf{r}_{\beta n'}). \tag{26}$$

In terms of the creation operators for the free fermion states introduced in Section 1 (which, in general, are not the exact eigenstates of h_α and h_β), the non-interacting part of the system Hamiltonian reads

$$H_\alpha = \sum_{\mathbf{k}_\alpha, \mathbf{k}'_\alpha} \langle \mathbf{k}'_\alpha | h_\alpha | \mathbf{k}_\alpha\rangle\, a^\dagger_{\mathbf{k}'_\alpha} a_{\mathbf{k}_\alpha} \tag{27}$$

$$H_\beta = \sum_{\mathbf{k}_\beta, \mathbf{k}'_\beta} \langle \mathbf{k}'_\beta | h_\beta | \mathbf{k}_\beta\rangle\, b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta}, \tag{28}$$

where the prefactors are given by

$$\langle \mathbf{k}'_\alpha | h_\alpha | \mathbf{k}_\alpha\rangle = \int d\mathbf{r}\, \langle \mathbf{k}'_\alpha|\mathbf{r}\rangle\, h_\alpha(\mathbf{r})\, \langle \mathbf{r}|\mathbf{k}_\alpha\rangle. \tag{29}$$

and similarly for $\langle \mathbf{k}'_\beta | h_\beta | \mathbf{k}_\beta\rangle$. In the same way, the two-body interacting parts of the Hamiltonian H read in second quantization, on this basis, as

$$V_{\alpha\alpha} = \frac{1}{2}\sum_{\mathbf{k}'_\alpha, \mathbf{k}_\alpha, \mathbf{q}'_\alpha, \mathbf{q}_\alpha} v_{\alpha\alpha}\begin{pmatrix} \mathbf{q}'_\alpha & \mathbf{q}_\alpha \\ \mathbf{k}'_\alpha & \mathbf{k}_\alpha \end{pmatrix} a^\dagger_{\mathbf{k}'_\alpha} a^\dagger_{\mathbf{q}'_\alpha} a_{\mathbf{q}_\alpha} a_{\mathbf{k}_\alpha} \tag{30}$$

$$V_{\alpha\beta} = \sum_{\mathbf{k}'_\alpha, \mathbf{k}_\alpha, \mathbf{k}'_\beta, \mathbf{k}_\beta} v_{\alpha\beta}\begin{pmatrix} \mathbf{k}'_\beta & \mathbf{k}_\beta \\ \mathbf{k}'_\alpha & \mathbf{k}_\alpha \end{pmatrix} a^\dagger_{\mathbf{k}'_\alpha} b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta} a_{\mathbf{k}_\alpha}, \tag{31}$$

where the prefactors are given by

$$\begin{aligned} v_{\alpha\alpha}\begin{pmatrix} \mathbf{q}'_\alpha & \mathbf{q}_\alpha \\ \mathbf{k}'_\alpha & \mathbf{k}_\alpha \end{pmatrix} &= \int d\mathbf{r}_\alpha\, d\mathbf{r}'_\alpha\, \langle \mathbf{k}'_\alpha|\mathbf{r}_\alpha\rangle\, \langle \mathbf{q}'_\alpha|\mathbf{r}'_\alpha\rangle\, v_{\alpha\alpha}(\mathbf{r}_\alpha, \mathbf{r}'_\alpha) \\ &\quad \times \langle \mathbf{r}'_\alpha|\mathbf{q}_\alpha\rangle\, \langle \mathbf{r}_\alpha|\mathbf{k}_\alpha\rangle \\ &= v_{\alpha\alpha}\begin{pmatrix} \mathbf{k}'_\alpha & \mathbf{k}_\alpha \\ \mathbf{q}'_\alpha & \mathbf{q}_\alpha \end{pmatrix}, \end{aligned} \tag{32}$$

and similarly for the other prefactors.

3.2 Orthogonal cobosons

3.2.1 "Creation potential"

Let us first consider cobosons forming an orthogonal set, these cobosons being not necessarily the exact one-pair eigenstates of the system Hamiltonian. Due to the closure relation (5) for orthogonal states, H acting on $|i\rangle$ then reads

$$H|i\rangle = \sum_m |m\rangle\, \langle m|H|i\rangle, \tag{33}$$

with $\langle m|H|i\rangle = E_i\, \delta_{m,i}$ if the cobosons are eigenstates of the system Hamiltonian, *i. e.*, if $(H - E_i)|i\rangle = 0$. Equation (33) leads to define the "creation potential" $V^\dagger_i$ for the coboson i as

$$[H, B^\dagger_i] = \sum_m \langle m|H|i\rangle\, B^\dagger_m + V^\dagger_i, \tag{34}$$

in order for the creation potential to be such that

$$V^\dagger_i |v\rangle = 0. \tag{35}$$

This insures $V^\dagger_i$ to indeed describe the interactions of the coboson i with the rest of the system. Let us now calculate this $V^\dagger_i$ explicitly.

(i) It is possible to split the commutator of $B_i^\dagger$, given in equation (4), with the part of the Hamiltonian acting on fermion pairs, into three terms:

$$\left[H_\alpha + H_\beta + V_{\alpha\beta}, B_i^\dagger\right] = A_1^\dagger + A_2^\dagger + A_3^\dagger, \tag{36}$$

with $A_1^\dagger$ in $a^\dagger b^\dagger$, $A_2^\dagger$ in $a^\dagger b^\dagger b^\dagger b$ and $A_3^\dagger$ in $a^\dagger b^\dagger a^\dagger a$. The first term $A_1^\dagger$ precisely reads

$$\begin{aligned} A_1^\dagger = &\sum_{\mathbf{k}'_\alpha,\mathbf{p}_\beta} a^\dagger_{\mathbf{k}'_\alpha} b^\dagger_{\mathbf{p}_\beta} \sum_{\mathbf{p}_\alpha} \langle \mathbf{k}'_\alpha | h_\alpha | \mathbf{p}_\alpha \rangle \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle \\ &+ \sum_{\mathbf{k}'_\beta,\mathbf{p}_\alpha} a^\dagger_{\mathbf{p}_\alpha} b^\dagger_{\mathbf{k}'_\beta} \sum_{\mathbf{p}_\beta} \langle \mathbf{k}'_\beta | h_\beta | \mathbf{p}_\beta \rangle \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle \\ &+ \sum_{\mathbf{k}'_\alpha,\mathbf{k}'_\beta} a^\dagger_{\mathbf{k}'_\alpha} b^\dagger_{\mathbf{k}'_\beta} \sum_{\mathbf{p}_\alpha,\mathbf{p}_\beta} v_{\alpha\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{p}_\beta \\ \mathbf{k}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle. \end{aligned} \tag{37}$$

By noting that H_α does not act on fermion β, while the prefactor of the last term is nothing but

$$v_{\alpha\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{p}_\beta \\ \mathbf{k}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} = \langle \mathbf{k}'_\beta, \mathbf{k}'_\alpha | V_{\alpha\beta} | \mathbf{p}_\alpha, \mathbf{p}_\beta \rangle,$$

it is easy to see that $A_1^\dagger$ can be rewritten in a compact form as

$$\begin{aligned} A_1^\dagger = &\sum_{\mathbf{k}'_\alpha,\mathbf{k}'_\beta} a^\dagger_{\mathbf{k}'_\alpha} b^\dagger_{\mathbf{k}'_\beta} \sum_{\mathbf{p}_\alpha,\mathbf{p}_\beta} \\ &\times \langle \mathbf{k}'_\beta, \mathbf{k}'_\alpha | H_\alpha + H_\beta + V_{\alpha\beta} | \mathbf{p}_\alpha, \mathbf{p}_\beta \rangle \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle. \end{aligned} \tag{38}$$

If we now use equation (6) to write $a^\dagger b^\dagger$ in terms of cobosons, we end, due to equation (2), with

$$A_1^\dagger = \sum_m \langle m | H | i \rangle B_m^\dagger, \tag{39}$$

which is just the first term of equation (34).
The second term on the RHS of equation (36), $A_2^\dagger$, appears as

$$\begin{aligned} A_2^\dagger = &\\ &\sum_{\mathbf{k}'_\alpha,\mathbf{p}_\beta} a^\dagger_{\mathbf{k}'_\alpha} b^\dagger_{\mathbf{p}_\beta} \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta} b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta} \sum_{\mathbf{p}_\alpha} v_{\alpha\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{k}_\beta \\ \mathbf{k}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle. \end{aligned} \tag{40}$$

We rewrite the first $a^\dagger b^\dagger$ in terms of coboson operators according to equation (6), to make $A_2^\dagger$ reading as

$$A_2^\dagger = \sum_m B_m^\dagger \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta} b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta} X_{\alpha\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i)$$

$$\begin{aligned} &X_{\alpha\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i) = \\ &\quad\sum_{\mathbf{k}'_\alpha,\mathbf{p}_\alpha,\mathbf{p}_\beta} \langle m | \mathbf{k}'_\alpha, \mathbf{p}_\beta \rangle v_{\alpha\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{k}_\beta \\ \mathbf{k}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle. \end{aligned} \tag{41}$$

In the same way, $A_3^\dagger$ is found to be

$$A_3^\dagger = \sum_m B_m^\dagger \sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha} a^\dagger_{\mathbf{k}'_\alpha} a_{\mathbf{k}_\alpha} Y_{\alpha\beta}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i)$$

$$\begin{aligned} &Y_{\alpha\beta}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i) = \\ &\quad\sum_{\mathbf{k}'_\beta,\mathbf{p}_\alpha,\mathbf{p}_\beta} \langle m | \mathbf{p}_\alpha, \mathbf{k}'_\beta \rangle v_{\alpha\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{p}_\beta \\ \mathbf{k}'_\alpha & \mathbf{k}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle. \end{aligned} \tag{42}$$

(ii) If we now turn to the interactions between fermions α, the same procedure leads to

$$\left[V_{\alpha\alpha}, B_i^\dagger\right] = \sum_m B_m^\dagger \sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha} a^\dagger_{\mathbf{k}'_\alpha} a_{\mathbf{k}_\alpha} Y_{\alpha\alpha}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i)$$

$$\begin{aligned} &Y_{\alpha\alpha}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i) = \\ &\quad\sum_{\mathbf{q}'_\alpha,\mathbf{p}_\alpha,\mathbf{p}_\beta} \langle m | \mathbf{q}'_\alpha, \mathbf{p}_\beta \rangle v_{\alpha\alpha} \begin{pmatrix} \mathbf{k}'_\alpha & \mathbf{k}_\alpha \\ \mathbf{q}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle, \end{aligned} \tag{43}$$

while the interactions between fermions β give

$$\left[V_{\beta\beta}, B_i^\dagger\right] = \sum_m B_m^\dagger \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta} b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta} X_{\beta\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i)$$

$$\begin{aligned} &X_{\beta\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i) = \\ &\quad\sum_{\mathbf{q}'_\beta,\mathbf{p}_\alpha,\mathbf{p}_\beta} \langle m | \mathbf{p}_\alpha, \mathbf{q}'_\beta \rangle v_{\beta\beta} \begin{pmatrix} \mathbf{k}'_\beta & \mathbf{k}_\beta \\ \mathbf{q}'_\beta & \mathbf{p}_\beta \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i \rangle \end{aligned} \tag{44}$$

(iii) By collecting the results of equations (36, 39, 41–44), the creation potential $V_i^\dagger$, defined in equation (34), finally reads

$$V_i^\dagger = \sum_m B_m^\dagger W_{mi},$$

where the operator W_{mi} is defined by

$$\begin{aligned} W_{mi} = &\sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha} a^\dagger_{\mathbf{k}'_\alpha} a_{\mathbf{k}_\alpha} [Y_{\alpha\alpha}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i) \\ &\qquad + Y_{\alpha\beta}(\mathbf{k}'_\alpha, \mathbf{k}_\alpha; m, i)] \\ &+ \sum_{\mathbf{k}'_\beta,\mathbf{k}_\beta} b^\dagger_{\mathbf{k}'_\beta} b_{\mathbf{k}_\beta} \left[X_{\beta\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i) + X_{\alpha\beta}(\mathbf{k}'_\beta, \mathbf{k}_\beta; m, i)\right]. \end{aligned} \tag{45}$$

Since $W_{mi}|v\rangle = 0$, it is thus easy to check that the condition (35) for a creation potential, is indeed fulfilled by this $V_i^\dagger$.

3.2.2 "Interaction scatterings"

The "direct interaction scatterings" between cobosons i and j physically come from the interactions between fermions (α, α), between fermions (β, β) and also from the interactions between fermions (α,β), with the part between the fermions (α,β) making the coboson i excluded. These interaction scatterings are formally defined through

$$\left[V_i^\dagger, B_j^\dagger\right] = \sum_{mn} \xi \begin{pmatrix} n & j \\ m & i \end{pmatrix} B_m^\dagger B_n^\dagger, \tag{46}$$

so that these scatterings are such that

$$\left[W_{mi}, B_j^\dagger\right] = \sum_{n} \xi \begin{pmatrix} n & j \\ m & i \end{pmatrix} B_n^\dagger. \tag{47}$$

To calculate ξ, let us consider the first term of equation (45), in $Y_{\alpha\alpha}$. Using equations (4, 6), the commutator of this first term with $B_j^\dagger$ reads

$$\begin{aligned}[W_{mi}^{(1)}, B_j^\dagger] &= \sum_{\mathbf{k}'_\alpha,\mathbf{k}_\alpha} \sum_{\mathbf{p}'_\alpha,\mathbf{p}'_\beta} Y_{\alpha\alpha}(\mathbf{k}'_\alpha,\mathbf{k}_\alpha; m,i) \\ &\quad \times \langle \mathbf{p}'_\beta, \mathbf{p}'_\alpha | j\rangle \, [a^\dagger_{\mathbf{k}'_\alpha} a_{\mathbf{k}_\alpha}, a^\dagger_{\mathbf{p}'_\alpha} b^\dagger_{\mathbf{p}'_\beta}] \\ &= \sum_{\mathbf{k}'_\alpha,\mathbf{p}'_\alpha,\mathbf{p}'_\beta} Y_{\alpha\alpha}(\mathbf{k}'_\alpha,\mathbf{p}'_\alpha; m,i)\, \langle \mathbf{p}'_\beta, \mathbf{p}'_\alpha | j\rangle \\ &\quad \times \sum_n B_n^\dagger \, \langle n|\mathbf{k}'_\alpha, \mathbf{p}'_\beta\rangle. \end{aligned} \tag{48}$$

By inserting $Y_{\alpha\alpha}$ given in equation (43) into the above equation, we find that the first term of $\xi \begin{pmatrix} n & j \\ m & i \end{pmatrix}$ reads

$$\begin{aligned}\xi_1 \begin{pmatrix} n & j \\ m & i \end{pmatrix} &= \sum_{\mathbf{k}'_\alpha,\mathbf{q}'_\alpha,\mathbf{p}'_\alpha,\mathbf{p}'_\beta,\mathbf{p}_\alpha,\mathbf{p}_\beta} \\ &\times \langle m|\mathbf{q}'_\alpha, \mathbf{p}_\beta\rangle \, \langle n|\mathbf{k}'_\alpha, \mathbf{p}'_\beta\rangle \, v_{\alpha\alpha} \begin{pmatrix} \mathbf{k}'_\alpha & \mathbf{p}'_\alpha \\ \mathbf{q}'_\alpha & \mathbf{p}_\alpha \end{pmatrix} \langle \mathbf{p}_\beta, \mathbf{p}_\alpha | i\rangle \, \langle \mathbf{p}'_\beta, \mathbf{p}'_\alpha | j\rangle. \end{aligned} \tag{49}$$

This $\xi_1 \begin{pmatrix} n & j \\ m & i \end{pmatrix}$ is shown in Figure 1b. It corresponds to an interaction between the fermions α of the "in" cobosons (i,j), the "out" cobosons (m,n) being made with the same fermion pair as the "in" cobosons. We can rewrite this ξ_1 in real space, by using equations (17) and (32) and by performing the summation over the various $\mathbf{k}$'s through closure relations. This leads to

$$\begin{aligned}\xi_1 \begin{pmatrix} n & j \\ m & i \end{pmatrix} &= \int d\mathbf{r}_{\alpha 1} d\mathbf{r}_{\alpha 2} d\mathbf{r}_{\beta 1} d\mathbf{r}_{\beta 2} \phi_m^*(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 1}) \phi_n^*(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 2}) \\ &\quad \times v_{\alpha\alpha}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\alpha 2}) \phi_i(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 1}) \phi_j(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 2}), \end{aligned} \tag{50}$$

which is shown in Figure 1c.

By calculating the contributions of the three other terms of equation (45) in the same way, we end with a direct interaction scattering which has a form very similar to the one for excitons [1], namely

$$\begin{aligned}\xi \begin{pmatrix} n & j \\ m & i \end{pmatrix} &= \int d\mathbf{r}_{\alpha 1}\, d\mathbf{r}_{\alpha 2}\, d\mathbf{r}_{\beta 1}\, d\mathbf{r}_{\beta 2}\, \phi_m^*(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 1}) \\ &\quad \times \phi_n^*(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 2})\, \phi_i(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 1})\, \phi_j(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 2}) \\ &\quad \times \left[v_{\alpha\alpha}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\alpha 2}) + v_{\beta\beta}(\mathbf{r}_{\beta 1}, \mathbf{r}_{\beta 2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 2}) \right. \\ &\quad \left. + v_{\alpha\beta}(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 1})\right]. \end{aligned} \tag{51}$$

This direct scattering is represented in Figure 2b: in it, no fermion exchange takes place between the "in" cobosons (i,j).

3.2.3 Matrix elements of the system Hamiltonian in the 2-coboson subspace

Using the commutators given in equations (34, 46), the Hamiltonian in the two-coboson subspace appears as

$$\begin{aligned}\langle v|B_m B_n\, H\, B_i^\dagger B_j^\dagger|v\rangle &= \\ &\sum_q \langle v|B_m B_n B_i^\dagger B_q^\dagger|v\rangle\, \langle q|H|j\rangle \;+\; (i \leftrightarrow j) \\ &+ \sum_{pq} \xi \begin{pmatrix} q & j \\ p & i \end{pmatrix} \langle v|B_m B_n B_p^\dagger B_q^\dagger|v\rangle. \end{aligned} \tag{52}$$

So that, using equation (22) for the scalar product of orthogonal cobosons, we end with

$$\begin{aligned}\langle v|B_m B_n\, H\, B_i^\dagger B_j^\dagger|v\rangle &= \Bigg[\delta_{m,i}\, \langle n|H|j\rangle + \delta_{n,i}\, \langle m|H|j\rangle \\ &- \sum_q \left(\lambda \left(\begin{smallmatrix} n & q \\ m & i \end{smallmatrix}\right) + \lambda \left(\begin{smallmatrix} m & q \\ n & i \end{smallmatrix}\right)\right) \langle q|H|j\rangle \\ &+ \xi \begin{pmatrix} n & j \\ m & i \end{pmatrix} - \xi^{\mathrm{in}} \begin{pmatrix} n & j \\ m & i \end{pmatrix} \Bigg] + [i \leftrightarrow j], \end{aligned} \tag{53}$$

where ξ^{in} is the exchange interaction scattering shown in Figure 2c. It is precisely given by

$$\begin{aligned}\xi^{\mathrm{in}} \begin{pmatrix} n & j \\ m & i \end{pmatrix} &= \sum_{pq} \lambda \left(\begin{smallmatrix} n & q \\ m & p \end{smallmatrix}\right) \xi \begin{pmatrix} q & j \\ p & i \end{pmatrix} \\ &= \int d\mathbf{r}_{\alpha 1}\, d\mathbf{r}_{\beta 1}\, d\mathbf{r}_{\alpha 2}\, d\mathbf{r}_{\beta 2}\, \langle m|\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 2}\rangle\, \langle n|\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 1}\rangle \\ &\quad \times \langle \mathbf{r}_{\beta 1}, \mathbf{r}_{\alpha 1}|i\rangle\, \langle \mathbf{r}_{\beta 2}, \mathbf{r}_{\alpha 2}|j\rangle \\ &\quad \times \left[v_{\alpha\alpha}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\alpha 2}) + v_{\beta\beta}(\mathbf{r}_{\beta 1}, \mathbf{r}_{\beta 2}) \right. \\ &\quad \left. + v_{\alpha\beta}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 1})\right]. \end{aligned} \tag{54}$$

In the case of coboson eigenstates of the system Hamiltonian, $\langle n|H|j\rangle = E_j\, \delta_{n,j}$, we readily recover the result for composite excitons [1].

3.3 Nonorthogonal cobosons

If the coboson states form a complete set for one-fermion-pair states, but if these states are not orthogonal, we expect the preceding results to be far more complicated. It turns out that, as for the scalar product of cobosons, given in equation (22), the only difference with the results for orthogonal cobosons comes from the naïve part.

3.3.1 Creation potential and interaction scattering

Let us briefly go through the same path as the one we used in the preceding subsection, the closure relation for cobosons being now given by equation (7).

The definition of the creation potential for the coboson i, given in equation (34) for orthogonal cobosons, now reads

$$\left[H, B_i^\dagger\right] = \sum_m B_m^\dagger \sum_q z_{mq}\, \langle q|H|i\rangle + V_i^\dagger, \tag{55}$$

in order to still have $V_i^\dagger|v\rangle = 0$. From the precise calculation of $V_i^\dagger$, we can then show, by using equation (9) to write $a^\dagger b^\dagger$ in terms of cobosons, that

$$V_i^\dagger = \sum_{mq} B_m^\dagger \, z_{mq} \, W_{qi} \qquad (56)$$

$$\left[V_i^\dagger, B_j^\dagger\right] = \sum_{mn} B_m^\dagger B_n^\dagger \sum_{pq} z_{mp} \, z_{nq} \, \xi \begin{pmatrix} q & j \\ p & i \end{pmatrix}, \qquad (57)$$

where W_{qi} is defined in equation (45) and ξ is the direct interaction scattering defined in equation (51).

3.3.2 Matrix elements of the system Hamiltonian in the 2-coboson subspace

Using equations (55, 57), we readily find

$$\langle v|B_m B_n H B_i^\dagger B_j^\dagger|v\rangle = \langle v|B_m B_n B_i^\dagger H B_j^\dagger|v\rangle \; + (i \leftrightarrow j) + \sum_{pqtu} \langle v|B_m B_n B_p^\dagger B_q^\dagger|v\rangle \, z_{pt} z_{qu} \, \xi \begin{pmatrix} u & j \\ t & i \end{pmatrix}. \qquad (58)$$

The closure relation (7) allows to write the first term of the above equation as

$$\langle v|B_m B_n B_i^\dagger H B_j^\dagger|v\rangle = \sum_{pq} \langle v|B_m B_n B_i^\dagger B_p^\dagger|v\rangle \, z_{pq} \, \langle q|H|j\rangle, \qquad (59)$$

so that, using the scalar product of coboson states given in equation (22), this first term reads

$$\langle v|B_m B_n B_i^\dagger H B_j^\dagger|v\rangle = \langle n|i\rangle \, \langle m|H|j\rangle + \langle m|i\rangle \, \langle n|H|j\rangle - \sum_{pq} \left[\lambda \left(\begin{smallmatrix} n & p \\ m & i \end{smallmatrix} \right) + \lambda \left(\begin{smallmatrix} m & p \\ n & i \end{smallmatrix} \right)\right] z_{pq} \, \langle q|H|j\rangle. \qquad (60)$$

If we now turn to the last term of equation (58), we find, from the same equation (22),

$$\sum_{pqtu} \langle v|B_m B_n B_p^\dagger B_q^\dagger|v\rangle \, z_{pt} z_{qu} \, \xi \begin{pmatrix} u & j \\ t & i \end{pmatrix} = \sum_{pqtu} \langle m|p\rangle \, \langle n|q\rangle \, z_{pt} \, z_{qu} \, \xi \begin{pmatrix} u & j \\ t & i \end{pmatrix} - \sum_{pqtu} \lambda \left(\begin{smallmatrix} n & q \\ m & p \end{smallmatrix} \right) \, z_{pt} \, z_{qu} \, \xi \begin{pmatrix} u & j \\ t & i \end{pmatrix} \; + \; (m \leftrightarrow n). \qquad (61)$$

The first term readily gives $\xi \begin{pmatrix} n & j \\ m & i \end{pmatrix}$, due to equation (8). By using the expressions of λ and ξ in $\mathbf{r}$ space given in equations (18, 51), the second term appears as

$$\sum_{pqtu} \lambda \left(\begin{smallmatrix} n & q \\ m & p \end{smallmatrix} \right) \, z_{pt} \, z_{qu} \, \xi \begin{pmatrix} u & j \\ t & i \end{pmatrix} = \sum_{pqtu} z_{pt} \, z_{qu} \int \{d\mathbf{r}\} \, d\{\mathbf{r}'\} \, \langle m|\mathbf{r}'_{\alpha 1}, \mathbf{r}'_{\beta_2}\rangle \, \langle n|\mathbf{r}'_{\alpha 2}, \mathbf{r}'_{\beta 1}\rangle \, \langle \mathbf{r}'_{\beta 1}\mathbf{r}'_{\alpha 1}|p\rangle \times \langle \mathbf{r}'_{\beta 2}, \mathbf{r}'_{\alpha 2}|q\rangle \, \langle t|\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 1}\rangle \, \langle u|\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 2}\rangle \times \big[v_{\alpha\alpha}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\alpha 2}) + v_{\beta\beta}(\mathbf{r}_{\beta 1}, \mathbf{r}_{\beta 2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha 1}, \mathbf{r}_{\beta 2}) + v_{\alpha\beta}(\mathbf{r}_{\alpha 2}, \mathbf{r}_{\beta 1}) \big] \, \langle \mathbf{r}_{\beta 1}, \mathbf{r}_{\alpha 1}|i\rangle \, \langle \mathbf{r}_{\beta 2}, \mathbf{r}_{\alpha 2}|j\rangle. \qquad (62)$$

The summations over (p,t) and (q,u) being performed through equation (7), we readily find that the sum (62) reduces to the exchange interaction scattering $\xi^{\rm in} \begin{pmatrix} n & j \\ m & i \end{pmatrix}$ given in equation (54).

The above results thus show that the matrix elements of the system Hamiltonian in an arbitrary two-coboson subspace are given by

$$\langle v|B_m B_n H B_i^\dagger B_j^\dagger|v\rangle = \left\{ \left[\langle n|i\rangle \, \langle m|H|j\rangle - \sum_{pq} \lambda \left(\begin{smallmatrix} n & p \\ m & i \end{smallmatrix} \right) \, z_{pq} \, \langle q|H|j\rangle \right] + [i \leftrightarrow j] + \xi \begin{pmatrix} n & j \\ m & i \end{pmatrix} - \xi^{\rm in} \begin{pmatrix} n & j \\ m & i \end{pmatrix} \right\} + \{m \leftrightarrow n\}. \qquad (63)$$

This result, which reduces to equation (53) when the coboson states are orthogonal, again shows that the part coming from interactions between cobosons is formally the same whatever is the complete set of states these cobosons form.

4 Many-body effects with arbitrary cobosons

The standard way to derive many-body effects between elementary quantum particles for which the system Hamiltonian splits as $H = H_0 + V$, goes through the iteration of

$$\frac{1}{a-H} = \frac{1}{a-H_0} + \frac{1}{a-H} \, V \, \frac{1}{a-H_0}. \qquad (64)$$

We have shown [2] that, in the case of composite excitons which are eigenstates of the semiconductor Hamiltonian, the equivalent of $H = H_0 + V$, deduced from $[H, B_i^\dagger] = E_i B_i^\dagger + V_i^\dagger$, is $H B_i^\dagger = (H + E_i) B_i^\dagger + V_i^\dagger$, so that the equivalent of equation (64) reads

$$\frac{1}{a-H} \, B_i^\dagger = B_i^\dagger \, \frac{1}{a-H-E_i} + \frac{1}{a-H} \, V_i^\dagger \, \frac{1}{a-H-E_i}. \qquad (65)$$

In the most general case considered in this work, the creation potential of the coboson i is defined through

$$[H, B_i^\dagger] = \sum_m B_m^\dagger \sum_q z_{mq} \, \langle q|H|i\rangle + V_i^\dagger = H_{ii} \, B_i^\dagger + v_i^\dagger + V_i^\dagger, \qquad (66)$$

where we have set

$$H_{mi} = \sum_q z_{mq} \, \langle q|H|i\rangle \quad \text{and} \quad v_i^\dagger = \sum_{m \neq i} B_m^\dagger \, H_{mi}.$$

$v_i^\dagger$ is such that $v_i^\dagger|v\rangle \neq 0$, while $[v_i^\dagger, B_j^\dagger] = 0$. The equivalent of equation (65) then reads

$$\frac{1}{a-H} \, B_i^\dagger = B_i^\dagger \, \frac{1}{a-H-H_{ii}} + \frac{1}{a-H} \, (v_i^\dagger + V_i^\dagger) \, \frac{1}{a-H-H_{ii}}. \qquad (67)$$

Using equation (67) which is not far more complicated than equation (65), we can follow the same procedure as the one used for excitons, to deduce the part of the many-body effects between arbitrary cobosons coming from interactions *between* the elementary fermions making these composite bosons. Their correlations read in terms of matrix elements between N-coboson states which look like

$$\langle v|B_{m_N}\cdots B_{m_1}\,\frac{1}{a-H}\,B_{i_1}^\dagger\cdots B_{i_N}^\dagger|v\rangle.$$

To calculate them, we first push $1/(a-H)$ to the right according to equation (67) and we eliminate the various "creation potentials" through equations (6) or (57). This makes appearing a lot of interaction scatterings ξ. We end with scalar products of N-coboson states, which do not contain the system Hamiltonian anymore. These scalar products are then calculated, as for excitons, in terms of Pauli scatterings between two cobosons, using equation (10) and equations (13) or (19) — as done to get equation (22) for just $N=2$ cobosons. When N is large, these scalar products are better represented by Shiva diagrams for fermion exchanges between P excitons, with $2\leq P\leq N$, as explained more in details in a forthcoming publication [11].

5 Conclusion

The present work shows how the concepts we have recently introduced to exactly treat the fermion exchanges which take place in the many-body physics of excitons, can be extended to any pair of fermions. The exact description of composite boson many-body effects relies on two sets of scatterings: the "Pauli scatterings" for fermion exchanges without interaction and the "interaction scatterings" for interaction without fermion exchange. To derive these scatterings, it is not necessary for the fermion pairs to be the exact eigenstates of the system Hamiltonian, nor to form an orthogonal set for one-pair states.

The present extension of the free exciton many-body theory to any kind of composite bosons is highly desirable for various problems in which the physically relevant composite bosons are not the system eigenstates: for example, all problems involving photons which are predominantly coupled to free excitons, while traps are present in the semiconductor sample, so that these free excitons are not the system eigenstates.

This many-body theory extended to any kind of cobosons is also going to be useful in the physics of "cold gases", since the composite nature of the atoms must play an important role in their many-body effects. It is commonly believed that atoms differ from excitons due to the very large mass difference which exists between fermions α and β. Excitons have a center-of-mass delocalized over the whole sample — or at least over their coherence length. On the opposite, atoms are commonly seen as highly localized objects. This is due to the very large atomic mass, the momentum dispersion of the associated wave packet, allowed by a finite temperature, increasing with the center-of-mass mass. As the reason for this localization disappears when the temperature decreases, ultracold atoms end by being delocalized over the whole sample. Due to this delocalization, they never are far away, so that fermion exchanges can take place between them. Just as in the case of excitons, the effects induced by fermion exchanges between atoms are as large as the ones due to interactions between the fermions making these atoms: Both effects scale as the atom density multiplied by the atom volume, provided that this product stays small compared to one, for the atoms to stay bound.

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