Universal quantum computation in decoherence-free subspaces with hot trapped ions

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We consider interactions that generate a universal set of quantum gates on logical qubits encoded in a collective-dephasing-free subspace, and discuss their implementations with trapped ions. This allows for the removal of the by-far largest source of decoherence in current trapped-ion experiments, collective dephasing. In addition, an explicit parametrization of all two-body Hamiltonians able to generate such gates without the system's state ever exiting the protected subspace is provided.

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I. INTRODUCTION

In quantum information processing tasks decoherence can be overcome either by an active approach or by a passive one. The former consists, in analogy with classical computation, of encoding information in a redundant fashion by means of the so-called error-correcting codes. In this approach information is encoded in subspaces of the total Hilbert space of the system in such a way that "errors" induced by the interaction with the environment can be detected and corrected without gaining information about the actual state of the system prior to corruption [1].

The passive approach, on the other hand, is an error preventing scheme, in which logical qubits are encoded within decoherence-free subspaces (DFS's), which do not decohere because of symmetry [2]. A simple example is provided by a system of N spins collectively interacting with the same reservoir, for which the interaction is mediated by the collective angular momentum raising and lowering operators $\hat{S}^{+}\equiv\Sigma_{i=1}^{N}\hat{S}_i^{+}$ and $\hat{S}^{-}\equiv\Sigma_{i=1}^{N}\hat{S}_i^{-}$, where $\hat{S}_i^{+}$ and $\hat{S}_i^{-}$ are the corresponding raising and lowering operators, respectively, of the ith particle. The collective operators have no support on the eigenstates of the total squared angular momentum $\hat{S}^2$ corresponding to zero eigenvalue. The evolution of these eigenstates is therefore unitary because they simply do not couple to the reservoir; and they can be used as a logical-qubit basis for decoherence-free quantum computation [3,4].

When the coupling to the environment is mediated by the collective z-angular-momentum operator $\hat{S}^z\equiv\Sigma_{i=1}^{N}\hat{S}_i^z$, the type of noise is called *collective dephasing*. The interaction Hamiltonian between the system and the bath is then proportional to $\hat{S}^z\otimes\hat{B}$, where $\hat{B}$ is an arbitrary operator acting on the Hilbert space associated to the bath. The action of this type of bath is equivalent to that of randomly fluctuating fields: a general qubit-state $|\Psi\rangle\equiv a|0\rangle+b|1\rangle$ transforms as $|\Psi\rangle\rightarrow a|0\rangle+be^{i\zeta}|1\rangle$, which leads to the loss of coherence of the state for ζ is a random fluctuating phase. By using one pair of physical qubits, whose members are labeled by the subindexes i_1 and i_2, to encode logical qubit i, one can protect information from the detrimental action of decoherence. In fact, the well-known [5,6] logical basis $B_{L_i}\equiv\{|0_{L_i}\rangle\equiv|0_{i_1}1_{i_2}\rangle;|1_{L_i}\rangle\equiv|1_{i_1}0_{i_2}\rangle\}$ spans a DFS protected against collective dephasing, which we call $\mathbb{V}_{DFS_{2_i}}$. That is, the logical state $|\Psi_{L_i}\rangle\equiv a|0_{L_i}\rangle+b|1_{L_i}\rangle=a|0_{i_1}\rangle|1_{i_2}\rangle+b|1_{i_1}\rangle|0_{i_2}\rangle$ evolves as $|\Psi_{L_i}\rangle\rightarrow a|0_{i_1}\rangle e^{i\zeta}|1_{i_2}\rangle+be^{i\zeta}|1_{i_1}\rangle|0_{i_2}\rangle=e^{i\zeta}(a|0_{L_i}\rangle+b|1_{L_i}\rangle)$ and is thus invariant up to an irrelevant global phase factor.

Two pairs of physical qubits, whose members are labeled by the subindexes i_1 and i_2, and j_1 and j_2, respectively, are in turn needed to encode two logical qubits i and j. The direct product subspace $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$, spanned by the basis $B_{L_i}\otimes B_{L_j}$, yields a DFS. However, one should note that this is not the total protected subspace supported by all four qubits if all four physical qubits experience the same phase fluctuations. In this case the states $|0_{i_1}0_{i_2}1_{j_1}1_{j_2}\rangle$ and $|1_{i_1}1_{i_2}0_{j_1}0_{j_2}\rangle$, which are outside $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$, are also protected against collective dephasing for they have the same amount of excitations as the states in $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$. In general, any coherent superposition of states with the same amount of excitations is immune against collective dephasing. Thus, the total protected subspace, which we call $\mathbb{V}_{DFS_{4_{ij}}}$, is that spanned by $B_{L_i}\otimes B_{L_j}$ together with the states $|0_{i_1}0_{i_2}1_{j_1}1_{j_2}\rangle$ and $|1_{i_1}1_{i_2}0_{j_1}0_{j_2}\rangle$. If pairs i and j are further apart than the typical noise correlation length—but with both qubits from each pair still subject to the same fluctuations—$\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$ is the only protected subspace.

On the experimental side, the demonstration of immunity of a DFS of two photons to collective noise was accomplished in [7] and realizations of DFS's for nuclear magnetic resonance (NMR) systems were carried out in [8]. The demonstration of a collective-dephasing-free quantum memory of one logical qubit composed of a pair of trapped $^9Be^+$ ions was first achieved in [9] and coherent oscillations between two logical states, encoded into the two Bell states $|\Psi_{\pm}\rangle=\frac{1}{\sqrt{2}}(|01\rangle\pm|10\rangle)$, by inducing a gradient of the magnetic field applied to both ions, were reported in [10,11]. Finally, entanglement lifetimes of more than 7 s [11] and robust entanglement lasting for more than 20 s [12] were attained using ground state hyperfine levels of $^9Be^+$ ions and ground state Zeeman sublevels of $^{40}Ca^+$ ions, respectively. These experiments demonstrated that for trapped ions collective dephasing is the major source of qubit decoherence. We therefore focus on this type of noise throughout the rest of the paper. Nevertheless, apart from the proof-of-principle experiments mentioned above, demonstrating the robustness of these subspaces, experimentally accessible implementations of DFS-encoded gates are still sparse; and in spite of the rich (but abstract) body of work on DFS's, a universal set of gates between two encoded logical qubits is yet to be demonstrated.

Proposals for ion trap quantum computing with DFS's exist, and they are essentially divided into two families that complement each other. In the first paradigm [6] gates between two logical qubits are implemented [6,13] by bringing together two pairs of ions (each pair encoding a logical qubit), initially stored in memory regions, to an interaction region where a simultaneous interaction among all four ions takes place according to the Sørensen-Mølmer (SM) gate described in [14,15]. Individual laser addressing is not necessary for this scheme, but a reliable ion-shuttling technique is an essential requirement. In addition, even though this scheme maps $\mathbb{V}_{DFS_{4_{ij}}}$ into itself, it does not preserve the state inside the DFS throughout the gate evolution [4]. The second paradigm [16,17] works in the individual laser addressing regime and relaxes the need of ion shuttling. In this approach, ions are trapped in a crystal-like effective potential created by arrays of multiconnected linear Paul traps. Each ion is associated to a neighbor to form a pair that encodes one logical qubit [17]. By inducing a $\hat{\sigma}^z$-dependent force (see [18–20], and references therein) on two ions, each from different pairs, it is in principle possible to implement a (geometric phase) $\hat{\sigma}^z$ gate between the logical qubits encoded into both pairs. Particular advantages of these $\hat{\sigma}^z$ gates are that they can be considerably fast and robust. It has been conjectured [19–21] though, that these $\hat{\sigma}^z$ gates are very inefficient with magnetic-field-insensitive (or "clock") states, which possess such remarkable coherence properties [11,22]. However, it would be very advantageous to combine clock states with DFS's as this would lead to very long coherence times and minimize the overhead due to quantum error correction.

In our present paper we assess different possible interactions involving only two physical qubits at a time that generate universal quantum gates on DFS-encoded qubits, and describe feasible experimental demonstrations of each of them with trapped ions. The work is conceptually divided into two parts. The first one (Sec. II) is devoted to the general formal classification of all two-body dynamics able to generate universal quantum gates inside the DFS without the system's state ever leaving it. The aim here is not to establish the set of formal conditions for a given Hamiltonian to generate universal DFS quantum computation, as in [3,4]; but rather to explicitly construct the allowed Hamiltonians in a simple way in terms of the Pauli operators associated to each physical qubit. This is to serve as a simple "classification table" for experimentalists to rapidly check whether the type of interactions present in their given system qualifies as a candidate for generating universal DFS quantum computation or not. In particular, we introduce the most general two-body Hamiltonian that generates universal quantum computation while guaranteeing the evolution to take place entirely inside $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$. Furthermore, we show that the only possible interaction between two logical qubits, which obeys the previous assumptions, is of the type $\hat{\sigma}^z \otimes \hat{\sigma}^z$. For the cases where leakage out of $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$ into $\mathbb{V}_{DFS_{4_{ij}}}$ is allowed, we consider the encoding recoupling scheme originally introduced in [25] for NMR systems. There, a maximally entangling gate is implemented on the DFS through a sequence of transformations that momentarily takes the composite state out of $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$ but never out of $\mathbb{V}_{DFS_{4_{ij}}}$.

The second part (Sec. III) describes the technical details of the implementation on trapped ions of the ideas presented in Sec. II. Our implementations work in the individual laser addressing regime and require no ion shuttling. We show that for the realization of local and conditional gates inside $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$, the SM gate and the $\hat{\sigma}^z$ gate, respectively, can be used. For the realization of the encoded recoupling scheme in turn, an alternative two-physical-qubit gate is required. The latter is based on bichromatic Raman fields and applies to all states in general, regardless of their magnetic properties, including clock states connected via dipole Raman transitions. Furthermore, this gate does not require the ions to be in their motional ground state, provided that they always remain in the Lamb-Dicke regime. Therefore, it is a potentially useful alternative to the SM gate and the $\hat{\sigma}^z$ gate also outside the context of DFS's. Our conclusions are finally summarized in Sec. IV.

II. GENERAL HAMILTONIANS FOR UNIVERSAL QUANTUM COMPUTATION IN THE DFS

A. Local operations: The logical SU(2) Lie algebra

We want to find a complete set of orthogonal operators mapping $\mathbb{V}_{DFS_{2_i}}$ (for any i) onto itself. We define then logical identity and Pauli operators, $\hat{\sigma}^0_{L_i} \equiv \hat{I}_{L_i}$, $\hat{\sigma}^1_{L_i} \equiv \hat{\sigma}^x_{L_i}$, $\hat{\sigma}^2_{L_i} \equiv \hat{\sigma}^y_{L_i}$, and $\hat{\sigma}^3_{L_i} \equiv \hat{\sigma}^z_{L_i}$ of the ith logical qubit, as

$$\hat{\sigma}^0_{L_i} \equiv \alpha_i \hat{\sigma}^0_{i_1} \otimes \hat{\sigma}^0_{i_2} - (1-\alpha_i)\hat{\sigma}^3_{i_1} \otimes \hat{\sigma}^3_{i_2} + \hat{0}_{L_i},$$

$$\hat{\sigma}^1_{L_i} \equiv \beta_i \hat{\sigma}^1_{i_1} \otimes \hat{\sigma}^1_{i_2} + (1-\beta_i)\hat{\sigma}^2_{i_1} \otimes \hat{\sigma}^2_{i_2} + \hat{0}_{L_i},$$

$$\hat{\sigma}^2_{L_i} \equiv \gamma_i \hat{\sigma}^2_{i_1} \otimes \hat{\sigma}^1_{i_2} - (1-\gamma_i)\hat{\sigma}^1_{i} \otimes \hat{\sigma}^2_{i_2} + \hat{0}_{L_i},$$

$$\hat{\sigma}^3_{L_i} \equiv \varepsilon_i \hat{\sigma}^3_{i_1} \otimes \hat{\sigma}^0_{i_2} - (1-\varepsilon_i)\hat{\sigma}^0_{i_1} \otimes \hat{\sigma}^3_{i_2} + \hat{0}_{L_i}, \tag{1}$$

where $\hat{\sigma}^p_{i_n}$ is the identity ($p=0$) or Pauli ($1 \le p \le 3$) operator associated to the nth ($n=1$ or 2) physical qubit of the ith pair, and with α_i, β_i, γ_i, and ε_i any real numbers such that $0 \le \alpha_i, \beta_i, \gamma_i$, and $\varepsilon_i \le 1$. The operator $\hat{0}_{L_i}$ represents the logical null operator, which is defined as any operator without support on $\mathbb{V}_{DFS_{2_i}}$. The operators in Eq. (1) map $\mathbb{V}_{DFS_{2_i}}$ onto itself and their action on B_{L_i} is exactly equivalent to that of the usual identity and Pauli physical operators on the computational basis. It can be seen that Eq. (1) is the most general way to construct them from the operators that act on the physical qubits. For example, if we added the term $\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^3_{i_2}$ to the definition of $\hat{\sigma}^3_{L_i}$ in Eq. (1) we would exit $\mathbb{V}_{DFS_{2_i}}$; terms as $\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^1_{i_2}$ would not take us out of the DFS but act like $\hat{\sigma}^1_{L_i}$ instead; and so on. Combinations as $\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^1_{i_2}-\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^2_{i_2}$ or $\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^3_{i_2}-i\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^0_{i_2}$ are allowed though, since they have no support on $\mathbb{V}_{DFS_{2_i}}$ and can therefore be grouped inside $\hat{0}_{L_i}$. In general there are sixteen possible products between $\hat{\sigma}^0_{i_1}$, $\hat{\sigma}^1_{i_1}$, $\hat{\sigma}^2_{i_1}$, and $\hat{\sigma}^3_{i_1}$, and $\hat{\sigma}^0_{i_2}$, $\hat{\sigma}^1_{i_2}$, $\hat{\sigma}^2_{i_2}$, and $\hat{\sigma}^3_{i_2}$. Each of these products, or combinations of them, apart from those already considered in Eq. (1), either takes the state out of $\mathbb{V}_{DFS_{2_i}}$, does not have the desired action, or has no support on $\mathbb{V}_{DFS_{2_i}}$ and is therefore absorbed inside the definition of $\hat{0}_{L_i}$. The most general expression for the logical null operator is given by

$$\begin{aligned}\hat{0}_{L_i} \equiv{}& \rho_i(\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^1_{i_2}-i\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^2_{i_2})+\theta_i(\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^0_{i_2}-i\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^3_{i_2})\\ &+\vartheta_i(\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^3_{i_2}-i\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^0_{i_2})+\zeta_i(\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^1_{i_2}-i\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^2_{i_2})\\ &+\kappa_i(\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^0_{i_2}+\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^3_{i_2})+\lambda_i(\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^1_{i_2}-\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^2_{i_2})\\ &+\varsigma_i(\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^2_{i_2}+\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^1_{i_2})+\xi_i(\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^0_{i_2}+\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^3_{i_2}),\end{aligned}\tag{2}$$

with ρ_i, θ_i, ϑ_i, ζ_i, κ_i, λ_i, ς_i, and ξ_i any complex numbers.

The operators in Eq. (1) are orthonormal: $\mathrm{Tr}[\hat{\sigma}^p_{L_i}\hat{\sigma}^q_{L_i}]=\delta_{pq}$, with p and $q=0,1,2$ or 3, and form therefore a complete orthonormal basis of the space of the complex operators acting on the two-dimensional subspace $\mathbb{V}_{DFS_{2_i}}$. They also satisfy, inside of $\mathbb{V}_{DFS_{2_i}}$, the desired SU(2) usual commutation relations: $[\hat{\sigma}^p_{L_i},\hat{\sigma}^q_{L_i}]=2i\epsilon_{pqr}\hat{\sigma}^r_{L_i}$, for p, q and $r=1$, 2, or 3; and $[\hat{\sigma}^0_{L_i},\hat{\sigma}^p_{L_i}]=0$, for $p=0$, 1, 2, or 3. As an example to show this, we calculate explicitly the commutator $[\hat{\sigma}^1_{L_i},\hat{\sigma}^2_{L_i}]$ and obtain

$$\begin{aligned}[\hat{\sigma}^1_{L_i},\hat{\sigma}^2_{L_i}]={}&2i(1-\beta_i-\gamma_i+2\beta_k\gamma_k)\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^0_{i_2}\\ &-2i(\beta_k+\gamma_i-2\beta_i\gamma_i)\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^3_{i_2}.\end{aligned}\tag{3}$$

Doing the identification $\varepsilon'_i\equiv1-\beta_i-\gamma_i+2\beta_i\gamma_i$ and since $0\le\beta_i\le1$ and $0\le\gamma_i\le1$ we see that $0\le\varepsilon'_i\le1$, which leads us to

$$[\hat{\sigma}^1_{L_i},\hat{\sigma}^2_{L_i}]=2i[\varepsilon'_k\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^0_{i_2}-(1-\varepsilon'_i)\hat{\sigma}^0_{i_1}\otimes\hat{\sigma}^3_{i_2}].\tag{4}$$

This is, inside of $\mathbb{V}_{DFS_{2_i}}$, exactly equivalent to $2i\hat{\sigma}^3_{L_i}$. Note that the logical operator obtained here and the fourth operator in Eq. (1) are actually not strictly equal, since ε_i and ε'_i are not necessarily the same number. Their difference, however, only shows when applied to states outside $\mathbb{V}_{DFS_{2_i}}$, their action on this subspace is exactly the same. All the other SU(2) fundamental commutation relations are straightforwardly obtained in the same way. We see thus that the logical Pauli operators defined in Eq. (1) are the most general representation of the SU(2) Lie algebra on $\mathbb{V}_{DFS_{2_i}}$ constructed from the physical-qubit operators.

We also notice that the logical operators $\bar{X}_i\equiv\frac{1}{2}(\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^1_{i_2}+\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^2_{i_2})$, $\bar{Y}_i\equiv\frac{1}{2}(\hat{\sigma}^2_{i_1}\otimes\hat{\sigma}^1_{i_2}-\hat{\sigma}^1_{i_1}\otimes\hat{\sigma}^2_{i_2})$ and $\bar{Z}_i\equiv\frac{1}{2}(\hat{\sigma}^3_{i_1}-\hat{\sigma}^3_{i_2})$ used in [13] are a particular case of Eq. (1), corresponding to $\beta_i=\gamma_i=\varepsilon_i=1/2$. These operators generate the SU(2) group on the whole Hilbert space, but they have the same action as those defined in Eq. (1) on $\mathbb{V}_{DFS_{2_i}}$. The advantage of the logical operators in Eq. (1) is that they give the experimentalist more freedom of choice, as any choice of α_i, β_i, γ_i, and ε_i works just as well in $\mathbb{V}_{DFS_{2_i}}$. As a matter of fact, we exploit this freedom below to simplify the procedure for obtaining DFS-encoded gates for trapped ions.

The situation is now completely equivalent to that of a physical qubit, with the logical states in B_{L_i} and logical operators in Eq. (1) playing the role of the physical ones. The important thing to keep in mind though is that these logical operators allow us to operate on the logical states in the same way as their physical counterparts without ever exiting $\mathbb{V}_{DFS_{2_i}}$. With this at hand we can now write down the Hamiltonian that generates the most general unitary operation on the ith logical qubit; it reads

$$\hat{H}_{L_i}\equiv B^0_i\hat{\sigma}^0_{L_i}+B^1_i\hat{\sigma}^1_{L_i}+B^2_i\hat{\sigma}^2_{L_i}+B^3_i\hat{\sigma}^3_{L_i},\tag{5}$$

with B^0_i, B^1_i, B^2_i, and B^3_i any real numbers (times arbitrary units of energy) that play the role of a "logical magnetic field." Notice that we are explicitly including the logical identity in Hamiltonian (5), even though it only introduces an irrelevant global phase factor. This is because we want to account, in the most general way, for the possibility of appearance of terms proportional to $\hat{\sigma}^3_{i_1}\otimes\hat{\sigma}^3_{i_2}$, which are not irrelevant for an implementation on physical qubits.

B. Computation in $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$: The two-physical-qubit interaction Hamiltonian

We proceed now with the interaction Hamiltonian between logical qubits i and j, $\hat{H}_{L_iL_j}$. Under the action of this Hamiltonian there can be no transfer of excitations between both qubit pairs, so that each logical qubit evolves inside its own encoded subspace. The only allowed interactions are then those ones composed of combinations of products of logical Pauli operators of both logical qubits. Nevertheless, the remarkable observation is that $\hat{\sigma}^3_{L_i}$ and $\hat{\sigma}^3_{L_j}$ are the only logical operators that do not involve interactions between the physical qubits from the same pair. Any product of two logical Pauli operators from both logical qubits other than $\hat{\sigma}^3_{L_i}\otimes\hat{\sigma}^3_{L_j}$ will necessarily contain products of more than two physical-qubit (nonidentity) operators. We see, therefore, that there exists only one type of two-body interaction able to generate nontrivial two logical qubit operations on the DFS and at the same time preserving the composite state always inside $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$. It is given by

$$\hat{H}_{L_iL_j} \propto \hat{\sigma}^3_{L_i} \otimes \hat{\sigma}^3_{L_j}. \tag{6}$$

This interaction between both logical qubits reduces to a simple Ising interaction between one physical qubit from pair i and one from j when the nonsymmetric choice ε_i and ε_j equal to 0, or 1, is taken. Also, the fact that the operators in the z direction play such a preferential role is not surprising, since, for collective dephasing, it is the total z angular momentum that mediates the coupling of the qubits to the environment; and our protected subspace is precisely that of null total z angular momentum.

The aim of Hamiltonians (5) and (6), together with expressions (1) for the single logical-qubit operators, is to provide a tool for the immediate classification of the allowed two-body dynamics for the implementation of DFS universal quantum computation. Any system whose Hamiltonian cannot be expressed as given by Eqs. (5) and (6), together with (1), is automatically excluded as a candidate for such computation, except, of course, for the possible appearance of any combination of physical-qubit operators that can be expressed as in Eq. (2).

C. Computation in $\mathbb{V}_{DFS_{4_{ij}}}$: The encoded recoupling scheme

An alternative technique to entangle logical qubits is the encoded recoupling scheme, which was originally developed for NMR systems in [25]. In this scheme, a $\hat{\sigma}^3 \otimes \hat{\sigma}^3$ interaction is effectively simulated by a sequence of $\hat{\sigma}^+ \otimes \hat{\sigma}^-$-type interactions between different physical qubits from both pairs. This provokes an actual transfer of excitations between both pairs, so that the logical qubits momentarily exit $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$ and "lose their encoded logical identity." But the total amount of excitations remains the same, so that the whole evolution takes place inside $\mathbb{V}_{DFS_{4_{ij}}}$. The technique is based on the identity

$$\begin{aligned} &e^{-i[\hat{\sigma}^+_{i_1}\otimes\hat{\sigma}^-_{j_1}+\text{H.c.}]\pi/4} e^{-i[\hat{\sigma}^+_{i_1}\otimes\hat{\sigma}^-_{i_2}+\text{H.c.}]\pi/2} [\hat{\sigma}^+_{i_2} \otimes \hat{\sigma}^-_{j_1} + \text{H.c.}] \\ &\quad\times e^{i[\hat{\sigma}^+_{i_1}\otimes\hat{\sigma}^-_{i_2}+\text{H.c.}]\pi/2} e^{i[\hat{\sigma}^+_{i_1}\otimes\hat{\sigma}^-_{j_1}+\text{H.c.}]\pi/4} \\ &\quad= \frac{1}{2}\hat{\sigma}^3_{i_2} \otimes (\hat{\sigma}^3_{j_1} - \hat{\sigma}^3_{i_1}). \end{aligned} \tag{7}$$

When applying this fivefold sequence of transformations to states in $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$, the product $\hat{\sigma}^3_{i_2} \otimes \hat{\sigma}^3_{i_1}$ on the right-hand side can be ignored, since it is proportional to the logical identity operator, and introduces thus nothing but a global phase factor. This leaves us with $\frac{1}{2}\hat{\sigma}^3_{i_2} \otimes \hat{\sigma}^3_{j_1}$, which is equivalent to $-\frac{1}{2}\hat{\sigma}^3_{L_i} \otimes \hat{\sigma}^3_{L_j} \equiv -\frac{1}{2}\hat{H}_{L_iL_j}$ [with the nonsymmetric choice $\varepsilon_i = 0$ and $\varepsilon_j = 1$ in Eq. (1)].

Also here only interactions between two physical qubits at a time are required, but the technique has the drawbacks that it requires more pulses and can be used only when pairs i and j experience the same phase fluctuations. Nevertheless, it constitutes an alternative to spin-dependent forces, especially when Ising-type interactions are not readily available, as appears to be the case with clock states connected via dipole Raman transitions.

III. IMPLEMENTATION ON TRAPPED IONS

We consider next N pairs of ions confined in a linear Paul trap, or in an arrangement of multiconnected linear Paul traps, where individual laser addressing is available. The collective vibrational mode along the axial direction z, of frequency ν, might be the center-of-mass or stretch mode. The ith logical qubit is encoded into a pair i of neighboring ions i_1 and i_2. We assume each ion i_n ($n = 1$ or 2) to have a mass M and an equilibrium position $z_{0_{i_n}}$. The ions may either possess three energy levels in a Λ configuration: two long-lived ground-state levels, and an excited electronic state; or two energy levels, one of which is a metastable state, and the other the ground state. In both cases, we label the physical qubit states as $|\uparrow_{i_n}\rangle \equiv |0_{i_n}\rangle$ and $|\downarrow_{i_n}\rangle \equiv |1_{i_n}\rangle$, and their internal transition frequency ω_0. For three-level ions the physical qubit states are encoded in the two long-lived ground-state levels; ω_0 is typically in the microwave region, and the qubit states are typically connected by a dipole Raman transition through the excited electronic state, driven by two laser beams A and B, of frequencies ω_A and ω_B and wave vectors along the z direction k_{A_z} and k_{B_z}. For two-level ions, in turn, the metastable state encodes $|\uparrow_{i_n}\rangle \equiv |0_{i_n}\rangle$, the ground state $|\downarrow_{i_n}\rangle \equiv |1_{i_n}\rangle$, and they are connected by a weak quadrupole optical transition directly driven by a single laser L, of frequency ω_L and wave vector along the z direction k_{L_z}.

A. Single-logical-qubit gates: $\hat{\sigma}^3_L$

We show first how to implement Hamiltonian (5) for the case $B^0_i = B^1_i = B^2_i = 0$. In this case it suffices to induce an ac Stark shift on only one of the members of the pair, for example, ion i_n, which can be done by the application of off-resonant fields δ detuned from the carrier transition. The interaction Hamiltonian in the interaction picture with respect to the unperturbed Hamiltonian without the laser field, and in the rotating wave approximation (RWA), with the condition $\omega_0 \gg \nu \gg \delta$, then reads $\hat{H}_{i_n} = \hbar\Omega_{i_n}\hat{\sigma}^+_{i_n} e^{i[\delta t + \varphi_{i_n}]} + \text{H.c.}$ Here Ω_{i_n} is the effective Rabi frequency coupling $|\uparrow_{i_n}\rangle$ with $|\downarrow_{i_n}\rangle$ and φ_{i_n} is the spin phase, the field's effective optical phase at position $z_{0_{i_n}}$.

From now on we will always work in the dispersive regime $|\Omega_{i_n}| \ll \delta$, in which perturbative calculations with $\frac{\Omega_{i_n}}{\delta}$ as a perturbation parameter are valid. In fact, a time-dependent second-order perturbative calculation, yields an effective time-independent Hamiltonian given by

$$\hat{H}_{i_n} = \hbar\frac{|\Omega_{i_n}|^2}{\delta}\hat{\sigma}^3_{i_n}. \tag{8}$$

Since, according to Eq. (1), $\hat{\sigma}^3_{L_i}$ coincides with $\hat{\sigma}^3_{i_1}$ for the nonsymmetric choice $\varepsilon_i = 1$ and with $-\hat{\sigma}^3_{i_2}$ for $\varepsilon_i = 0$, it is

$$\hat{H}_{i_n} = B^3_i\hat{\sigma}^3_{L_i}, \tag{9}$$

with $B^3_i \equiv \pm\hbar\frac{|\Omega_{i_n}|^2}{\delta}$, the "+" ("−") sign corresponding to $n = 1$ ($n = 2$); implementing thus the desired logical Hamiltonian.

It is important to notice that in the above derivation, as well as in the rest of the paper, the resolved-sideband limit

$|\Omega_{i_n}| \ll \nu$ is assumed. In this regime, by tuning the laser frequency, it is always possible to select the stationary terms of the Hamiltonian and to neglect—in the RWA—all other terms rotating at the different vibrational modes' frequencies. This was exploited here to neglect terms involving any vibrational mode frequency by setting ω_L (or $\omega_A-\omega_B$) close to ω_0, and is exploited in the next subsections to select the desired vibrational mode by setting it close to resonance with a sideband transition to such a mode.

B. Single-logical-qubit gates: $\hat{\sigma}_L^{\phi}$

We now concentrate on the implementation of Hamiltonian $\hat{H}_{i_1i_2}=C_i\hat{\sigma}_{L_i}^{\phi_i}$, where C_i is a constant and $\hat{\sigma}_{L_i}^{\phi_i}$, defined as $\hat{\sigma}_{L_i}^{\phi_i}\equiv\cos(\phi_i)\hat{\sigma}_{L_i}^1+\sin(\phi_i)\hat{\sigma}_{L_i}^2\equiv e^{-i\phi_i}\hat{\sigma}_{L_i}^{+}+e^{i\phi_i}\hat{\sigma}_{L_i}^{-}$, is the operator contained in the equatorial plane of the logical Bloch sphere with azimuth angle ϕ_i. This is equivalent to Hamiltonian (5) with $B_i^0=B_i^3=0$, $B_i^1\equiv C_i\cos(\phi_i)$, and $B_i^2\equiv C_i\sin(\phi_i)$. For any fixed value of ϕ_i the ability to implement such Hamiltonian, together with Hamiltonian (9), suffices to generate any SU(2) operation on the ith logical qubit.

In this case it is possible to use the SM gate [14,19], driven by one field detuned by δ from the red sideband, plus another one detuned by $-\delta$ from the blue one. Here we show nonetheless that only one of these fields suffices as long as one remains in $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$. We extend the ideas of Ref. [27] and consider a laser field irradiating simultaneously both ions of the ith pair. When the laser frequency or laser frequency difference is close to resonance with a sideband transition, a coupling between the internal qubit states and the relevant vibrational mode is possible. We choose the first red sideband transition for definiteness, but the blue one would work just as well. That is, we set $\omega_A-\omega_B=\omega_0-\nu-\delta$ and $\Delta k_z\equiv k_{B_z}-k_{A_z}\neq 0$ (noncopropagating beams is a further requirement for Raman couplings), or $\omega_L=\omega_0-\nu-\delta$. All other vibrational modes can be neglected under the RWA because we are in the resolved-sideband limit and they give no stationary contribution. The Lamb-Dicke parameter is defined as $\eta_\nu\equiv\Delta k_z\frac{z_\nu}{\sqrt{2N}}\equiv\frac{1}{\sqrt{2N}}\Delta k_z\sqrt{\frac{\hbar}{M\nu}}$, or $\eta_\nu\equiv k_{L_z}\frac{z_\nu}{\sqrt{2N}}\equiv\frac{1}{\sqrt{2N}}k_{L_z}\sqrt{\frac{\hbar}{M\nu}}$, where z_ν is the root-mean-square width of the motional ground-state wave packet. We assume next that the system is in the Lamb-Dicke limit (LDL) $\eta_\nu^2(n_\nu+1/2)\ll 1$, with n_ν the mean phonon population, meaning that the wave packet is very localized as compared to the fields' wavelengths $2\pi\Delta k_z^{-1}$ or $2\pi k_{L_z}^{-1}$. In this case the interaction Hamiltonian in the RWA is given by $\hat{H}_{i_1i_2}=\hbar[\Omega_{i_1}\hat{\sigma}_{i_1}^{+}(e^{i\nu t}+i\eta_\nu\hat{a}_\nu)e^{i(\delta t+\varphi_{i_1})}+\Omega_{i_2}\hat{\sigma}_{i_2}^{+}(e^{i\nu t}-i\eta_\nu\hat{a}_\nu)e^{i(\delta t+\varphi_{i_2})}+\text{H.c.}]$, where $\hat{a}_\nu$ is the annihilation operator of one phonon. Notice that here, in spite of being in the resolved sideband limit, we have not neglected the fast oscillating term proportional to $e^{i\nu t}$, since for very low values of η_ν the contribution of the latter might be comparable to that of the stationary term proportional to η_ν.

Taking both Rabi frequencies equal, $\Omega_{i_1}=\Omega_{i_2}\equiv\Omega_i$, yields the time-independent effective Hamiltonian:

$$\hat{H}_{i_1i_2}=\hbar\frac{|\Omega_i|^2}{\nu+\delta}(\hat{\sigma}_{i_1}^z+\hat{\sigma}_{i_2}^z)+\hbar\frac{|\Omega_i\eta_\nu|^2}{\delta}[\hat{I}+(\hat{\sigma}_{i_1}^z+\hat{\sigma}_{i_2}^z)(\hat{n}_\nu+1/2)-(\hat{\sigma}_{i_1}^{+}\otimes\hat{\sigma}_{i_2}^{-}e^{i(\varphi_{i_1}-\varphi_{i_2})}+\text{H.c.})], \tag{10}$$

with $\hat{n}_\nu\equiv\hat{a}_\nu^\dagger\hat{a}_\nu$. The identity operator $\hat{I}$ can be omitted as it only generates an irrelevant global phase factor; and so can the terms proportional to $\hat{\sigma}_{i_1}^z+\hat{\sigma}_{i_2}^z$, for they are equivalent to $\hat{0}_{L_i}$ [taking $\rho_i=\zeta_i=\vartheta_i=\theta_i=\kappa_i=\lambda_i=\varsigma_i=0$ in Eq. (2)]. We thus see that in $\mathbb{V}_{DFS_{2_i}}$ Hamiltonian (10) is equivalent to

$$\begin{aligned}\hat{H}_{i_1i_2}&=-\hbar\frac{|\Omega_i\eta_\nu|^2}{\delta}[\hat{\sigma}_{i_1}^{+}\otimes\hat{\sigma}_{i_2}^{-}e^{i\phi_i}+\text{H.c.}]\\&=-\hbar\frac{|\Omega_i\eta_\nu|^2}{2\delta}\{\cos(\phi_i)[\hat{\sigma}_{i_1}^1\otimes\hat{\sigma}_{i_2}^1+\hat{\sigma}_{i_1}^2\otimes\hat{\sigma}_{i_2}^2]\\&\quad+\sin(\phi_i)[\hat{\sigma}_{i_1}^2\otimes\hat{\sigma}_{i_2}^1-\hat{\sigma}_{i_1}^1\otimes\hat{\sigma}_{i_2}^2]\},\end{aligned} \tag{11}$$

with $\phi_i\equiv\varphi_{i_1}-\varphi_{i_2}$. A direct exchange of quanta between both ions through a virtual excitation of the vibrational mode. It is in turn immediate to express Eq. (11) as the desired Hamiltonian

$$\hat{H}_{i_1i_2}=C_i\hat{\sigma}_{L_i}^{\phi_i}, \tag{12}$$

with $C_i\equiv-\hbar\frac{|\Omega_i\eta_\nu|^2}{\delta}$, and where $\beta_i=\gamma_i=1/2$ have been taken in Eq. (1).

C. Two logical-qubit gates

A $\hat{\sigma}^3\otimes\hat{\sigma}^3$-type interaction between physical qubits from different pairs is required to realize Hamiltonian (6). However, interaction schemes such as the one described in the previous subsection that use the vibrational mode as a virtual mediator always involve products as $\hat{\sigma}^{\pm}\otimes\hat{\sigma}^{\pm}$. So a $\hat{\sigma}^3\otimes\hat{\sigma}^3$ effective interaction, with no explicit dependence on $\hat{a}_\nu$ or $\hat{a}_\nu^\dagger$, appears only as a fourth-order contribution, negligible as compared to the contributions from previous orders. Therefore, it is very inefficient to realize a nonlocal gate between two logical qubits using only two-body interactions, with no explicit dependence on the vibrational operators, under the requirement that the states involved in the operation stay in the encoded subspace $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$. If, on the other hand, the vibrational mode is allowed to be actually populated, instead of just being used as a virtual mediator, optical forces that exert a state-dependent force onto the ions can be used to generate effectively such an interaction [18].

As the implementation of these optical forces is described elsewhere [18–20,23,24], we show here how to implement the alternative encoded recoupling scheme. This requires $\hat{\sigma}^{+}\otimes\hat{\sigma}^{-}$-type interactions between different physical qubits from both pairs to realize the sequence of transformations in Eq. (7). Such a sequence of pulses momentarily takes the states out of $\mathbb{V}_{DFS_{2_i}}\otimes\mathbb{V}_{DFS_{2_j}}$ but never out of $\mathbb{V}_{DFS_{4_{ij}}}$. This implies that the bichromatic gate described in the previous subsection cannot be used here, since the two terms proportional to $\hat{\sigma}_{i_1}^z+\hat{\sigma}_{i_2}^z$ eliminated from Hamiltonian (10) because of being proportional to $\hat{0}_{L_i}$ do have a finite support on

$\mathbb{V}_{DFS_{4_{ij}}}$. Hamiltonian (10) is not equivalent to Eq. (11) outside $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$.

The first term in Eq. (10) is only a Stark shift that contains no interaction between both physical qubits. From the formal point of view one could simply leave it in the evolution and then undo its action at the end by just applying local pulses. An experimentally accessible approach to compensate for it is, either to perform an effective qubit frequency renormalization [19], or to use compensation-laser techniques [26]. We therefore disregard it, which leaves us with

$$\hat{H}_{i_1 i_2} = \hbar \frac{|\Omega_i \eta_\nu|^2}{\delta}[\hat{I} + (\hat{\sigma}^z_{i_1} + \hat{\sigma}^z_{i_2})(\hat{n}_\nu + 1/2) - (\hat{\sigma}^+_{i_1} \otimes \hat{\sigma}^-_{i_2} e^{i\phi_i} + \text{H.c.})], \tag{13}$$

still containing the second term proportional to $\hat{n}_\nu + 1/2$. This term entangles the internal and motional degrees of freedom, so that unless the system is previously cooled to, and kept in, its motional ground state $n_\nu = 0$, it makes the action of the gate explicitly dependent on the vibrational state. In order to circumvent this we notice that adding Hamiltonian (13) to itself, with the replacements $\delta \leftrightarrow -\delta$ and $\phi_i \leftrightarrow \phi_i + \pi$, yields exactly twice Hamiltonian (11) without an explicit dependence on the vibrational operators, even when applied to any two ions from different pairs and for states outside $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$.

Now, the latter is exactly the effective Hamiltonian of the system when, simultaneously with the field so far considered, a second field is applied on both ions. Since the two ions can now be any ions from any pair we drop the subindex i. The second field (herein labeled with a "$\sim$") must have the same base Rabi frequency $\tilde{\Omega} = \Omega$ and Lamb-Dicke parameter $\tilde{\eta}_\nu = \eta_\nu$ as the first one; must be exactly π radians out of phase with it, $\tilde{\phi} = \phi + \pi$; and must be $-\delta$ detuned from the same sideband transition: $\omega_L^{\sim} = \omega_0 - \nu + \delta$.

This bichromatic scheme differs from the SM gate in an important way: the latter is based on Raman beams detuned from opposite sidebands rather than the same sideband, as proposed here. The SM gate can operate in both the dispersive regime $|\Omega_{k_n}| \ll \delta$, in which the vibrational degree of freedom is also only used as a virtual mediator, and the "fast" regime of small δ—more naturally described as a $\hat{\sigma}^\phi$-dependent force—in which the motional degree of freedom is actually populated during the gate evolution. Nevertheless, in both regimes the SM-gate Hamiltonian includes terms as $\hat{\sigma}^+ \otimes \hat{\sigma}^+$ and $\hat{\sigma}^- \otimes \hat{\sigma}^-$, which are undesired in this context for, even though they do not have support on $\mathbb{V}_{DFS_{2_i}} \otimes \mathbb{V}_{DFS_{2_j}}$, they take some states out of $\mathbb{V}_{DFS_{4_{ij}}}$. The SM gate is therefore not useful for the implementation of the encoded recoupling scheme.

Since the encoded recoupling scheme involves several pulses, the duration of the procedure must be compared to realistic entanglement lifetimes. For instance, taking the experimental values at the Innsbruck experiment [12]: $\Omega = 2\pi \times 100$ kHz, $\eta_\nu = 0.0165$, and $\delta = 2\pi \times 16.5$ kHz $(10 \times \eta_\nu \Omega)$, the time required to realize, for instance, the pulse $e^{-i[\hat{\sigma}^+ \otimes \hat{\sigma}^- + \text{H.c.}]\pi/2}$ is $\tau \equiv \pi\delta/2(\Omega\eta_\nu)^2 = 3$ ms, which is four orders of magnitude smaller than the 20 s robust entanglement reported there. We also note that in the case of Raman transitions the effective Lamb-Dicke parameter η_ν is typically larger, yielding a considerable speedup.

In addition, we have numerically simulated the pulse sequence (7) to generate a logical π-phase gate. The model used for the simulation is that of the usual Jaynes-Cummings Hamiltonian only under the optical RWA, and where the terms in its Taylor expansion of order higher than 2 in the Lamb-Dicke parameter where neglected. Two $\pi/2$ pulses on the logical qubits were inserted just before and after the phase gate to turn it into a logical controlled-NOT (CNOT) gate that, written in terms of the physical-qubit states, has the following truth table:

$$\begin{aligned}
|0_{i_1}1_{i_2},0_{j_1}1_{j_2}\rangle &\to |0_{i_1}1_{i_2},0_{j_1}1_{j_2}\rangle,\\
|0_{i_1}1_{i_2},1_{j_1}0_{j_2}\rangle &\to |0_{i_1}1_{i_2},1_{j_1}0_{j_2}\rangle,\\
|1_{i_1}0_{i_2},0_{j_1}1_{j_2}\rangle &\to |1_{i_1}0_{i_2},1_{j_1}0_{j_2}\rangle,\\
|1_{i_1}0_{i_2},1_{j_1}0_{j_2}\rangle &\to |1_{i_1}0_{i_2},0_{j_1}1_{j_2}\rangle.
\end{aligned} \tag{14}$$

Figure 1 shows the numerically calculated evolution of selected populations during the CNOT operation. The total required pulse area is 5π and thus the total required time for a CNOT is approximately 15 ms. For all four test cases fidelities exceeding 90% are calculated. We used the values from above for the laser settings. The motional mode frequency was chosen to be $2\pi \times 1.2$ MHz, a typical value in the Innsbruck experiments. Decoherence effects such as magnetic field fluctuations and laser frequency fluctuations are not considered because the evolution takes place predominantly in the DFS. The assumed addressing error of 5% on adjacent ions reduces the fidelities by about 3% and off-resonant excitations produce a 3% error. Other decoherence sources, like intensity fluctuations, motional heating, etc., are expected to contribute not significantly. The errors due to off-resonant excitations can be greatly reduced by pulse shaping, i.e., switching laser pulses adiabatically as compared to the Rabi frequencies. Addressing errors can be reduced considerably with composite pulse sequences such that they appear only in second order. Thus we estimate that the total infidelities of the proposed scheme is potentially well below 1% even with present technology, so that the gate fulfills the requirements set in [28] for fault-tolerant quantum computation. We note, however, that for useful quantum computation, higher gate fidelities than estimated here reduce the overhead dramatically.

D. Phase sensitivity

Let us briefly discuss the sensitivity of the protocol to fluctuations of the optical phase of the driving fields due to relative path instabilities, which can be a serious limiting factor for the fidelity of the gates [15,19]. For the implementation of single-qubit operations in the DFS [Hamiltonians (9) and (12)] on qubits using optical transitions this will not represent a major problem, since copropagating laser beams

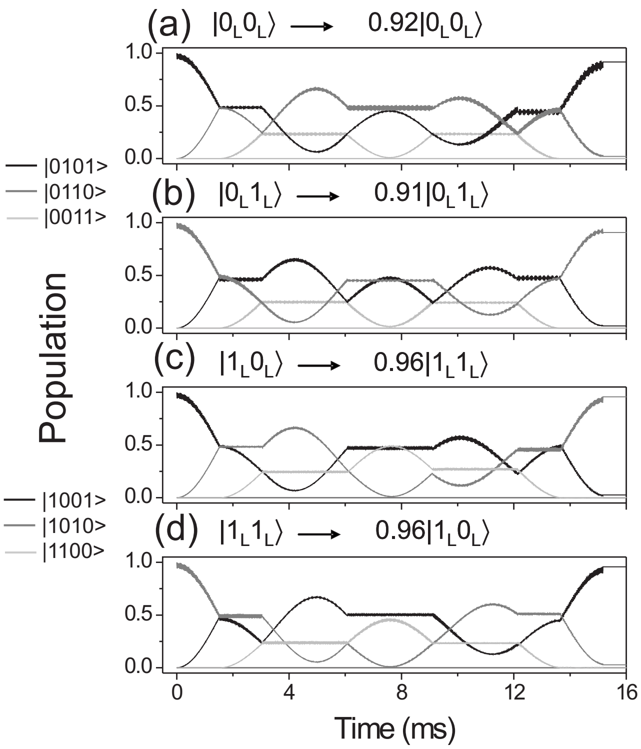

FIG. 1. State evolution of two logical qubits encoded in four physical qubits under the action of a CNOT gate implemented with the encoded recoupling scheme for four different input states. While in (a) and (b) mainly the physical states $|0101\rangle$, $|0110\rangle$, $|0011\rangle$ are populated, in (c) and (d) mostly $|1001\rangle$, $|1010\rangle$, $|1100\rangle$ are involved. The presence of off-resonant excitations leads to very rapid micro-oscillations in the populations that, in this resolution, simply appear as thicker lines. Pulse shaping can be used to suppress the imperfections induced by those off-resonant excitations.

can be used and thus relative phase fluctuations can be made quite small. A very similar situation arises for Raman-driven qubits as each pair of noncopropagating laser beams acts simultaneously on neighboring ions and can be viewed as one effective field, with phase fluctuations between both ions small as in the above case. We thus conclude that path length differences can be efficiently controlled in the single logical qubit case.

We now turn to the two-logical-qubit gates. In contrast to single-qubit operations, here each bichromatic beam acts simultaneously on two ions that are not necessarily neighbors. Nevertheless, even for ion spacings of up to 1 mm the beams take essentially the same path and thus, e.g., relative fluctuations of the air's refraction index are not significant. Furthermore, since the fivefold pulse sequence (7) yields a π-phase gate, which does not depend on the spin phase, interferometric stability is required only throughout the pulse sequence. Therefore long-term interferometric stability is not necessary.

IV. CONCLUSION

We considered the different interactions involving only two physical qubits at a time that generate universal quantum gates on collective-dephasing-free-encoded qubits, and described feasible experimental demonstrations of each of them with trapped ions using existing technology. A general formal classification of all two-body dynamics able to generate such gates without the system's state ever leaving the encoded subspace was provided in terms of the Pauli operators associated to each physical qubit, together with the explicit presentation of the allowed Hamiltonians. The implementation of these Hamiltonians operates in the individual laser addressing regime and requires no ion shuttling, so that it complements the collective-ion-addressing-based proposals. Also, no ground-state cooling is needed, provided that the ions always remain in the Lamb-Dicke regime. In addition, it makes use of a two-ion gate based on bichromatic Raman fields that can be applied to clock states connected via dipole Raman transitions. Finally, even though this gate is particularly well suited for implementing universal quantum computing in DFS's, it constitutes by itself a potentially useful alternative to other entangling gates outside the context of DFS's.

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