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Is an ion string laser-cooled like a single ion?

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Abstract

We discuss laser cooling of a string of two-level atoms confined in a linear ion trap. The equations for the dynamics are first analysed in the Lamb–Dicke regime (i.e. when the size of the mechanical wavefunction is much smaller than the laser wavelength), and then in general, for Doppler cooling. In the latter case, using several consecutive steps of averaging we derive from the full quantum mechanical master equation an equation for the total mechanical energy of the one-dimensional crystal, defined on a coarse-grained energy scale whose grid size is smaller than the linewidth of the electronic transition. This equation describes the cooling dynamics for an arbitrary number of ions in the quantum regime. We discuss the analogy and the scalability of the string's laser-cooling dynamics with that of a single trapped ion.

1. Introduction

Coulomb crystals have been the object of investigation in several fields of physics [1] and have experienced a renewed interest from the field of quantum information processing with trapped ions [2, 3]. The necessity of producing cold ion strings has motivated the investigation of laser cooling of this many-body system. Laser cooling allows for dissipation of the ions' thermal motional energy, and at sufficiently low temperature the ions assume an ordered configuration that minimizes their potential energy [1]. In this limit the residual motion of the ions is described by the normal modes of the crystal, like in crystalline states observed in larger systems. Ion strings are one-dimensional Coulomb crystals realized in linear Paul traps with steep radial confinement [4]: here the cold ions line up along the axial direction (trap axis). The mechanical effect of light on these systems is rather complex due to the large number of motional degrees of freedom. Nevertheless, each ion interacts individually with the light field, since collective effects in the scattering are usually negligible. This makes the general principle

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of the interaction simple: it is the recoil of the individual ion after absorption or emission of a photon which excites the collective modes of the string through the ion displacement.

On the basis of theoretical investigations it has been argued that an ion crystal is cooled like a single ion, i.e. that the dynamics of cooling are qualitatively the same [5]. This statement has been disputed for three-dimensional Coulomb crystals in Paul traps, where non-linear effects appear due to the fast rf-drive [6]. Nevertheless, this statement may well apply to three-dimensional crystals in Penning traps and to ion strings in linear Paul traps, where no rf-drive is present on the trap axis. Here, we present and discuss the results of some recent works on laser cooling of ion strings [7–9] in view of the question of analogy and scalability with the dynamics of laser cooling of a single ion. We consider the motion along the string, but the arguments can be easily extended to the radial motion.

This contribution is organized as follows. In section 2 we introduce the model and the corresponding equations for studying the dynamics. In section 3 we discuss the dynamics in the Lamb–Dicke regime, i.e. when the motional wavepacket is well localized on the scale of the laser wavelength. In section 4, starting from general considerations, we show the fundamental steps and discuss the approximations leading to an equation for the total mechanical energy of the crystal. The scope of this contribution is to give an overview of the results and the conclusions of some recent papers [7–9]. The reader interested in the theoretical details is referred to these publications.

2. The model

We consider a string of N ions of mass m, confined in a trap of axial frequency ν. Their axial motion is described by the normal modes with frequencies $\nu_1, \ldots, \nu_N$, which scale with the trap frequency ν. The corresponding Hamiltonian has the form

$$H_{\rm mec} = \sum_{\alpha=1}^{N} \hbar \nu_\alpha (a_\alpha^\dagger a_\alpha + \tfrac{1}{2}), \tag{1}$$

where $a_\alpha^\dagger, a_\alpha$ are the creation and annihilation operators of a vibrational quantum $\hbar\nu_\alpha$. For later convenience, we introduce the eigenstates $|\boldsymbol{n}\rangle$ of (1), with $\boldsymbol{n} = (n_1, \ldots, n_N)$. They satisfy the eigenvalue equation $H_{\rm mec}|\boldsymbol{n}\rangle = E_{\boldsymbol{n}}|\boldsymbol{n}\rangle$ with $E_{\boldsymbol{n}} = \sum_\alpha \hbar\nu_\alpha(n_\alpha + 1/2)$. The spectrum of $H_{\rm mec}$ does not exhibit the discreteness and equispacing of the single harmonic oscillator spectrum, but shows a dense distribution of levels, as the ratios between the mode frequencies $\nu_1, \ldots, \nu_N$ are incommensurate. At sufficiently high energy and large N the spectrum assumes a quasicontinuum character, such that the density of states is well approximated by a smooth function $g(E) \propto E^{N-1}$, as shown in figure 1 for the case $N = 3$.

In this representation, the displacement q_j of an ion from its classical equilibrium position $x_j^{(0)}$ in the string is the superposition of the normal mode displacements, and is given by $q_j = \sum_\alpha x_{0,\alpha} b_\alpha^j (a_\alpha^\dagger + a_\alpha)$, where $x_{0,\alpha} = \sqrt{\hbar/2m\nu_\alpha}$ is the size of the ground state wavepacket for the oscillator at frequency ν_α, and b_α^j is a scalar which gives the dependence of the mode displacement on the position of the ion in the chain.

The internal degrees of freedom of each ion are treated as a two-level dipole transition at frequency ω_0 and dipole moment $\boldsymbol{d}$. The transition is resonantly driven by a laser, represented here by the classical field $\boldsymbol{E}(x,t) = E_0\vec{\epsilon}\cos(\omega_L t - k_x x)$ with frequency ω_L, amplitude E_0, and polarization $\vec{\epsilon}$. The wavevector component along the $\hat{x}$-axis is $k_x = |\vec{k}|\cos\theta_0$, where $|\vec{k}|$ is the modulus of the wavevector and θ_0 is the angle between laser and motional axis. In the dipole approximation the interaction of the ion at x_j with the light is described by the potential $V = \boldsymbol{d}\cdot\boldsymbol{E}(x_j, t)$, and the explicit dependence of V on the position x enters through the terms

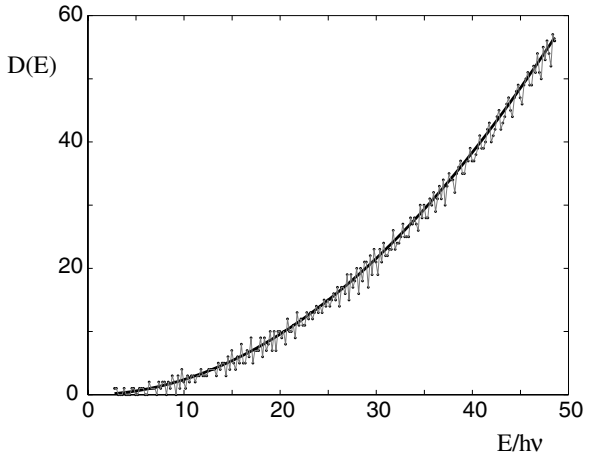

Figure 1. The number of states $D(E)$ as a function of the total energy $E/\hbar\nu$ evaluated for three ions on a grid $\Delta E = \hbar\nu/5$ (dots joined by the grey line); smoothed function $g(E)\Delta E$ (see text) with $N = 3$ (black curve). The frequencies of the modes are $\nu_1 = \nu$, $\nu_2 = 1.7321\nu$, $\nu_3 = 2.4083\nu$ (from [9]).

$\mathrm{e}^{\pm \mathrm{i}k_x x_j} = \mathrm{e}^{\pm \mathrm{i}k_x (x_j^{(0)} + q_j)}$. This operator is responsible for the mechanical effect of the light on the ion string. It also shows that by driving a single ion of the string, the whole collective motion can be excited.

The master equation for the density matrix ρ describing the full dynamics of the internal and motional degrees of freedom has the form [9]

$$\partial_t \rho = \frac{1}{\mathrm{i}\hbar}[H_{\mathrm{mec}}, \rho] + \sum_{j=1}^{N}\left(\frac{1}{\mathrm{i}\hbar}[H_j(q_j), \rho] + \mathcal{L}_j(q_j)\rho\right). \tag{2}$$

Here H_j is the Hamilton operator for the dynamics of the ion at $x_j^{(0)}$, coupled to the laser with Rabi frequency $\Omega_j = |\boldsymbol{d} \cdot \vec{\epsilon} E_0|/2\hbar$ and detuning $\Delta = \omega_L - \omega_0$ from resonance, and $\mathcal{L}_j$ is the Liouvillian which describes spontaneous decay at rate γ. The operators H_j and $\mathcal{L}_j$ contain the displacement q_j of the ion through the operator $\mathrm{e}^{\mathrm{i}k' q_j}$, where $k' = k_x$ in $H_j(q_j)$, and $k' = |k| \cos\theta$ in the Liouvillian $\mathcal{L}_j$. The latter describes the recoil due to spontaneous emission of a photon at an angle θ with the motional axis. The probability of emission along θ is the dipole pattern of spontaneous emission, given by the probability distribution $\mathcal{N}(\cos\theta)$ [10].

The number of degrees of freedom can be reduced by considering the case when the dipoles are driven below saturation. This regime allows for a perturbative expansion in the parameter Ω_j/γ, and the excited states of the dipole transitions are adiabatically eliminated at second order in this expansion. This procedure removes the internal degrees of freedom from the equations for the motional degrees of freedom. The dynamics of the reduced density matrix μ for the motion in the basis $\{|\boldsymbol{n}\rangle\}$ are determined by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{n}|\mu|\boldsymbol{m}\rangle = &-\frac{\mathrm{i}}{\hbar}(E_n - E_m)\langle \boldsymbol{n}|\mu|\boldsymbol{m}\rangle \\ &- \sum_j\left[\sum_l (L^j_{l\to n}\langle \boldsymbol{l}|\mu|\boldsymbol{m}\rangle + (L^j_{l\to m})^*\langle \boldsymbol{n}|\mu|\boldsymbol{l}\rangle) + \sum_{r,s}\tilde{L}^j_{r,s\to n,m}\langle \boldsymbol{r}|\mu|\boldsymbol{s}\rangle\right].\end{aligned} \tag{3}$$

The coefficients $L^j_{l\to n}$, $\tilde{L}^j_{r,s\to n,m}$ are of order Ω_j^2/γ and describe spontaneous Raman scattering processes (and coherences between such processes), i.e. transitions of the system from state

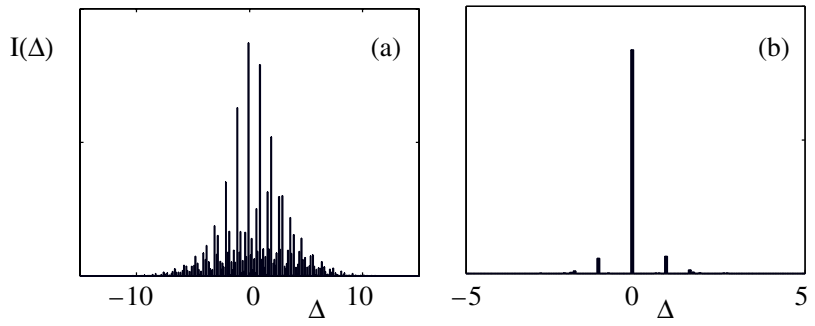

Figure 2. Absorption spectrum for a string of two ions, as obtained by scanning the laser frequency across the dipole transition and recording the level of fluorescence. It is given by $I(\Delta) = \sum_{(E_n - E_l)/\hbar = \Delta} |\langle \boldsymbol{n}| \exp(\mathrm{i}kx)|\boldsymbol{l}\rangle|^2 P(\boldsymbol{n})$. The detuning Δ is in units of ν and $I(\Delta)$ is plotted on a grid of width $\nu/10$ (note that here the natural line-broadening has not been considered). $P(\boldsymbol{n})$ is a thermal distribution with average energy per mode $\langle \bar{E} \rangle = 7.5\hbar\nu$. The Lamb–Dicke parameter $\eta = \sqrt{\omega_R/\nu}$ is in (a) $\eta = 0.6$ and in (b) $\eta = 0.1$. In (b) the carrier and the blue and red sidebands are visible, while in (a) many transitions appear where the vibrational numbers change simultaneously in both modes, or by more than one in a single mode (from [7]).

$|\boldsymbol{n}\rangle$ to state $|\boldsymbol{n}'\rangle$ when the ion at $x_j^{(0)}$ scatters a photon. The structure of these coefficients is the product of two probability amplitudes. One is associated with absorption of a laser photon along the transition $|g, \boldsymbol{n}\rangle \to |e, \boldsymbol{k}\rangle$, and its dependence on the energies E_n, E_k of the participating states is a Lorentzian with width $\hbar\gamma$ and peaked in $E_n - E_k = \hbar\Delta$,

$$W(E_n - E_k) \propto \frac{1}{E_n - E_k - \hbar\Delta - \mathrm{i}\hbar\gamma/2}. \tag{4}$$

The other is the Franck–Condon overlap between the motional states involved in the process of absorption/emission of a photon, and for the absorption of a photon by the ion at $x_j^{(0)}$ and the states $|\boldsymbol{n}\rangle$, $|\boldsymbol{k}\rangle$ it has the form

$$F_{n,k}^{k'} = \langle \boldsymbol{n}|\mathrm{e}^{\mathrm{i}k'q_j}|\boldsymbol{k}\rangle. \tag{5}$$

Its specific form depends on the quantum mechanical details of the states, i.e. on the vibrational numbers. Nevertheless, its average properties describe a distribution with centre at the energy shifted from E_n by the recoil energy $\hbar\omega_R$, and with variance around this value given by the Doppler width $\sqrt{2\hbar\omega_R E_n/N}$. Here we have introduced the recoil frequency $\omega_R = \hbar k^2/2m$. Figures 2(a) and (b) display the absorption spectrum of two ions as a function of the laser detuning Δ and for two different ratios ω_R/ν: the height of the vertical bars is determined by the Franck–Condon overlap between the states participating to the scattering, the width of the distribution by their average properties.

Three energy scales can now be identified which characterize the dynamics in (3): the trap frequency ν which scales the mode frequencies and thus the motion of the string, the linewidth γ of the dipole transition, which determines the states participating in the scattering process, and the recoil frequency ω_R which scales the mechanical energy exchanged by absorption/emission of a photon.

3. The Lamb–Dicke regime

When the atomic motional wavepacket is well localized with respect to the laser wavelength (Lamb–Dicke regime), the Doppler linewidth is much smaller than the mode frequencies.

In this limit, the Franck–Condon coefficients in (5) can be approximated by their first-order expansion in the Lamb–Dicke parameter η, where $\eta = k\sqrt{\hbar/2m\nu}$ [7, 8]. In this hierarchy, the transition at zero order is the carrier, where the motional state of the string is not changed by absorption/emission of a photon. To first order, the vibrational numbers of the motional state are changed by precisely one phonon, i.e. according to $|\boldsymbol{n}\rangle \to |\boldsymbol{n}+\mathbf{1}_\beta\rangle$ for the 'blue sideband' transition, and $|\boldsymbol{n}\rangle \to |\boldsymbol{n}-\mathbf{1}_\beta\rangle$ for the 'red sideband' transition. Here $\beta = 1, \ldots, N$ and the vector $\mathbf{1}_\beta$ is defined through $(\mathbf{1}_\beta)_\alpha = \delta_{\alpha,\beta}$ for $\alpha = 1, \ldots, N$. In this limit, transitions where the vibrational numbers of two or more modes change simultaneously, or where one mode gains or loses more than one phonon, are of higher order in η and thus neglected. This behaviour is reflected in the absorption spectrum as shown in figure 2(b). In equation (3) the coefficients L, $\tilde{L}$ scale with powers of η. In the Lamb–Dicke limit, the coupling of populations to coherences between different sideband transitions can be neglected, as the coupling itself is much smaller than the self-energy of the coherences, i.e. $\eta^2\Omega_j^2/\gamma \ll |\nu_\beta - \nu_\alpha|$. Then, equation (3) reduces to the rate equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \boldsymbol{n}|\mu|\boldsymbol{n}\rangle = -\langle \boldsymbol{n}|\mu|\boldsymbol{n}\rangle \sum_{j=1}^{N}\sum_{\beta=1}^{N}(\tilde{L}^j_{\boldsymbol{n}\to\boldsymbol{n}+\mathbf{1}_\beta} + \tilde{L}^j_{\boldsymbol{n}\to\boldsymbol{n}-\mathbf{1}_\beta}) + \sum_{j=1}^{N}\sum_{\beta=1}^{N}[\langle \boldsymbol{n}+\mathbf{1}_\beta|\mu|\boldsymbol{n}+\mathbf{1}_\beta\rangle \tilde{L}^j_{\boldsymbol{n}+\mathbf{1}_\beta\to\boldsymbol{n}} + \langle \boldsymbol{n}-\mathbf{1}_\beta|\mu|\boldsymbol{n}-\mathbf{1}_\beta\rangle \tilde{L}^j_{\boldsymbol{n}-\mathbf{1}_\beta\to\boldsymbol{n}}], \quad (6)$$

where $\tilde{L}^j_{\boldsymbol{n}\pm\mathbf{1}_\beta\to\boldsymbol{n}}$ are real and positive coefficients, and the subscript $\boldsymbol{n}\pm\mathbf{1}_\beta \to \boldsymbol{n}$ shortens the subscript $(\boldsymbol{n}\pm\mathbf{1}_\beta, \boldsymbol{n}\pm\mathbf{1}_\beta) \to (\boldsymbol{n}, \boldsymbol{n})$, as defined in equation (3)[4]. This result implies that each mode of an ion string is cooled independently from the others, like a mode of a single ion in a trap [8]. Thus, in the Lamb–Dicke limit concepts for laser cooling of single ions [10] can be applied to the individual modes of the string. This result holds both for resolved-sideband cooling and for Doppler cooling (non-resolved sideband cooling) [8, 9], as well as for other ground-state laser-cooling techniques implemented for single ions [11].

4. An ergodic equation for the string's dynamics

Outside the Lamb–Dicke regime it is difficult to apply the perturbative expansions which have been developed for the study of the dynamics of single ions [10]. Nevertheless, in the limit $\gamma > \nu_1, \ldots, \nu_N$ (Doppler cooling) it is possible to examine the crystal's motional energy on a coarse-grained energy scale, and thus to single out the quantities determining the cooling dynamics, while other quantities which depend on the fine details of the quantum states appear in the equations only through their expectation values [9].

We focus on the dynamics of the crystal's total mechanical energy, which is the quantity of interest when Doppler cooling the string. For this purpose, we analyse the dynamics between energy shells of fixed width ΔE, which comprise all the states of the Hilbert space whose energies fall into the interval $[E, E + \Delta E]$, as shown in figure 3 for the case of two modes. The shell width ΔE is chosen by requiring that the function (4) is practically constant over ΔE, which implies $\Delta E \ll \hbar\gamma$. At sufficiently high energy the number of states within a shell is large enough to be well approximated by the quantity $g(E)\Delta E$. In this limit, and when a sufficiently large number of pairs of states is coupled between two shells (like in the situation illustrated in figure 2(a)), the coupling of the shell population to coherences within and between shells is efficiently averaged out. In other words, the dynamics of the shell

[4] Note that at this order in the Lamb–Dicke parameter $\tilde{L}^j_{\boldsymbol{n}\to\boldsymbol{n}+\mathbf{1}_\beta} + \tilde{L}^j_{\boldsymbol{n}\to\boldsymbol{n}-\mathbf{1}_\beta}$ is the rate out of the state $\boldsymbol{n}$.

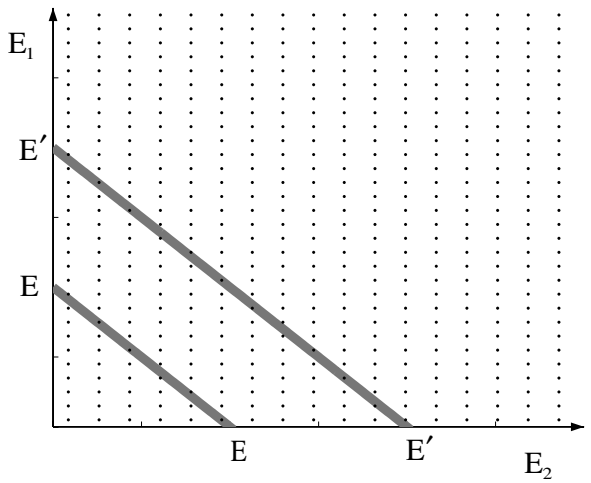

Figure 3. Coarse-grained energy space for the case of two modes of frequency ν_1, ν_2. The points are the states with energy $E_{\boldsymbol{n}} = E_{n_1} + E_{n_2}$. The broad lines represent two energy shells at energies E and E' (from [9]).

populations is negligibly affected by the coherences. The negligible role of coherences in the dissipative dynamics has been confirmed in numerical calculations for the case of two ions in [7]. The validity of this approximation relies on the quasi-continuum property of the spectrum. In this limit the coupling between two shells, which in principle depends on the Franck–Condon overlaps between the individual quantum states, can be approximated by its average value. This is expressed by

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{l}{}'|\langle \boldsymbol{l}|\mathrm{e}^{-\mathrm{i}k'q_j}|\boldsymbol{n}\rangle|^2 \approx \Delta E g(E_l) Q^{(k')}(E_{\boldsymbol{n}}, E_l), \tag{7}$$

where $\sum_l'$ denotes the sum over all motional states within a shell at energy E_l, and Q is a function of the shell energies: it describes the average coupling between two states in the shells at energies $E_{\boldsymbol{n}}$ and E_l due to photon scattering. Assumption (7) corresponds to neglecting the dependence of the LHS on the details of the state $|\boldsymbol{n}\rangle$. We substitute the discrete sums with integrals by defining the population densities $P(E, t)$ of the energy shells:

$$\Delta E P(E, t) = \sum_{\boldsymbol{n}}{}'\langle \boldsymbol{n}|\rho|\boldsymbol{n}\rangle, \tag{8}$$

where $E_{\boldsymbol{n}}$ is within the energy interval $[E, E + \Delta E]$. The density $P(E, t)$ is normalized, $\int_0^\infty \mathrm{d}E\, P(E, t) = 1$. Thus, we arrive at a rate equation for the motional energy with the form [9]

$$\frac{\mathrm{d}}{\mathrm{d}t}P(E, t) = -P(E, t)\int_0^\infty \mathrm{d}E_1\, f(E \to E_1) + \int_0^\infty \mathrm{d}E_1\, \tilde{f}(E_1 \to E)P(E_1, t), \tag{9}$$

where the coefficients f, $\tilde{f}$ are given by

$$f(E \to E_1) = \gamma\frac{\Omega^2}{4}N\frac{g(E_1)Q^{(k_z)}(E, E_1)}{[(E_1 - E)/\hbar - \Delta]^2 + \gamma^2/4},$$

$$\tilde{f}(E_1 \to E) = \int_{-1}^{1}\mathrm{d}(\cos\theta)\,\mathcal{N}(\cos\theta)g(E)\int_0^\infty \mathrm{d}E_2\, Q^{(k\cos\theta)}(E_2, E)f(E_1 \to E_2).$$

The first term in equation (9) describes transitions out of the shell, while the second term describes transitions into the shell, resulting from the absorption of a photon and its spontaneous emission averaged over all angles θ with the motional axis.

Equation (9) is an equation for the string's mechanical energy and it is the main result of this derivation. The coefficients Q in f, $\tilde{f}$ are known through their moments, which can be evaluated from the Franck–Condon coefficients. They scale with the number N of ions (modes): equation (9) describes the laser-cooling dynamics of a single ion in the limit of a quasi-continuum spectrum of mechanical energies, i.e. for $\gamma \gg \nu$, so that the approximations discussed above apply. The explicit form of the coefficients can be evaluated in several limits. In what follows we discuss the limit $\omega_R \ll \gamma$. Other limiting cases can be found in [9].

For $\omega_R \ll \gamma$ the variation of the population $P(E)$ on the energy scale $\hbar\omega_R$ is small, i.e. $\hbar\omega_R|\partial P(E,t)/\partial E| \ll P(E,t)$. By expanding around E to first order in the parameter ω_R/γ, we get a Fokker–Planck equation of the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}P(E,\tau) = -\frac{\partial}{\partial E}[A(E)P(E,\tau)] + \frac{\partial^2}{\partial E^2}\left[\frac{E}{N}P(E,\tau)\right], \tag{10}$$

where $\tau = 4M\Omega^2\gamma\omega_R(\cos^2\theta_0 + \alpha)L(0)t$ is a rescaled time, with $L(0) = 1/(4\Delta^2 + \gamma^2)$, $\alpha = \int_{-1}^{+1} \mathrm{d}\cos\theta \cos^2\theta \mathcal{N}(\cos\theta)$, and $A(E) = 1 + 2E\Delta L(0)\cos^2\theta_0/N(\cos^2\theta_0 + \alpha)$. We have introduced the parameter M which scales the time τ: it represents the (effective) number of ions in the string which are driven by the laser, assuming that the laser intensity is approximately the same for all driven ions. The cooling rate, characterizing the time at which the steady-state is reached, is now given by [9]

$$\Gamma_{\mathrm{cool}} = \frac{M}{N}\frac{8\omega_R\cos^2\theta_0\Omega^2\gamma|\Delta|}{(4\Delta^2+\gamma^2)^2}, \tag{11}$$

and it increases linearly with the number of driven ions M. Thus the cooling rate is maximum if all ions are driven, as expected below saturation. The steady-state energy is found to be

$$\langle E\rangle = N\gamma\frac{\alpha + \cos^2\theta_0}{4\cos^2\theta_0}\left(\frac{\gamma}{2|\Delta|} + \frac{2|\Delta|}{\gamma}\right) \tag{12}$$

which contains the dependence of the cooling limit on the angle θ_0 between the cooling laser beam and the direction of the motion. The final energy is minimal for $\cos\theta_0 = 1$, i.e. when the laser propagates parallel to the motional axis (that here corresponds to the trap axis), and it diverges for $\cos\theta_0 = 0$, when the laser is orthogonal to the trap axis and there is no cooling.

The minimum of the final energy versus detuning is reached for $\Delta = -\gamma/2$, as in the case of one ion (see for example [10]). Inserting into (12) $N = 1$, $\alpha = 1/3$ (which corresponds to spatially isotropic spontaneous emission) and $\cos\theta_0 = 1$, we find the same result as Javanainen and Stenholm [12] in their semiclassical expansion for one ion. Hence, the final energy for N ions is N times the steady state energy achieved by Doppler cooling of one ion. This general result has also been found earlier in special cases, such as a Coulomb cluster in the Lamb–Dicke regime [5], and a two-ion crystal treated in [13] by extending the method of [12].

Since the results we have shown so far agree precisely with those found earlier in specific cases, the general procedure which leads us from the quantum mechanical equations to the rate equation for the energy can be considered the common basis which underlies and unifies these earlier treatments. Furthermore, we have shown that the final energy of an N-ion crystal is N times that of a single motional mode, while the cooling rate scales with the fraction of driven ions.

The assumptions leading to the Fokker–Planck equation (10) were equation (7) and the slow variation of $P(E)$ on the scale of the recoil energy. The latter condition corresponds

to the second-order expansion in $\hbar$ of [12]. In that work the derivation of a Fokker–Planck equation was based on the limit of overdamped oscillation, $\gamma \gg \nu$, in order to adiabatically eliminate the excited state from the equations. Our derivation, however, does not necessarily imply this limit. Only in the case of $N = 1$ ion one must have $\Delta E \gg \hbar\nu$ in order to fulfil the condition of a large density of states, $D(E) \gg 1$, and thus for this special case the overdamped oscillator limit is a requirement for the validity of (10).

5. Conclusions and discussion

We have presented and discussed laser cooling of an ion string in view of the analogy with laser cooling of a single ion. We have focused on laser cooling in the Lamb–Dicke regime and on Doppler cooling in the general case, where a perturbative treatment is not possible. Here, we have shown the main steps of the derivation that lead to an equation for the string's mechanical energy, where the number of ions enters as a scaling factor. The results evaluated in the limiting cases of this equation agree with the ones obtained in previous treatments, which were developed in perturbation theory strictly valid in the limiting cases alone [5, 10, 12, 13]. Our results, with marginal changes, can also be applied when the radial motion of the string is taken into account. In that case, the dimensionality enters into the coefficients of the energy rate equation through the number of modes, which for a crystal of N ions is $3N$, and through the spatial distribution of the scattered photons.

On the basis of these results one can assert that an ion string is cooled like a single ion. The limits of validity of this statements are found when coherences between different motional states play a relevant role in the dynamics, for example in presence of exact degeneracies [7], or of additional interactions between the ions such as dipole–dipole coupling [14].

To conclude, we have modelled a complex (mesoscopic) system of N ions and N modes starting from the equations for the individual quantum states, and we have derived an equation for the dynamics of its total energy. Other incoherent processes in physical systems can be studied in an analogous way, provided that the rate determining the dynamics of interest can be singled out, and that on the corresponding energy scale the spectrum of energy levels is a quasi-continuum.

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